# Order of Accuracy for Non-Smooth Solution

Hiroaki Nishikawa

January 27, 2016

## 1 Order of Accuracy

Consider the advection equation in one dimension:

$$a\partial_x u = f, \tag{1}$$

where $a$ is a constant advection speed and $f = f(x)$. Suppose we discretize it on a uniform grid of spacing $h$ by a finite-difference scheme:

$$a\frac{3u_j - 4u_{j-1} + u_{j-2}}{2h} = f_j, \tag{2}$$

where $u_j$, $u_{j-1}$, and $u_{j-2}$ are the numerical solution values at nodes $j$, $j-1$ and $j-2$, respectively, and $f_j = f(x_j)$. The order of accuracy is found from the truncation error of the scheme. To derive the truncation error, we expand the exact solution in Taylor series around the node $j$ based on the assumption that the exact solution is smooth (i.e., differentiable), and substitute it into the difference scheme, resulting in the following expression:

$$a(\partial_x u)_j = f_j + \frac{h^2}{3}(\partial_{xxx}u)_j + O(h^3). \tag{3}$$

Since the exact solution satisfies $a\partial_x u = f$ at $j$, we are left with the truncation error:

$$\mathcal{TE}_j = \frac{h^2}{3}(\partial_{xxx}u)_j + O(h^3) = O(h^2). \tag{4}$$

Therefore, the order of accuracy of the above scheme is second order. The error in the numerical solution (the discretization error), whose order is known to agree with the truncation error order on regular grids, is also second order. An interesting question is: What if the exact solution is not smooth? Well, clearly, the above analysis is not valid for non-smooth solutions because it violates the assumption that the exact solution is smooth and can be expanded in Taylor series. Then, another interesting question would be: What would be the order of accuracy observed numerically in grid refinement?

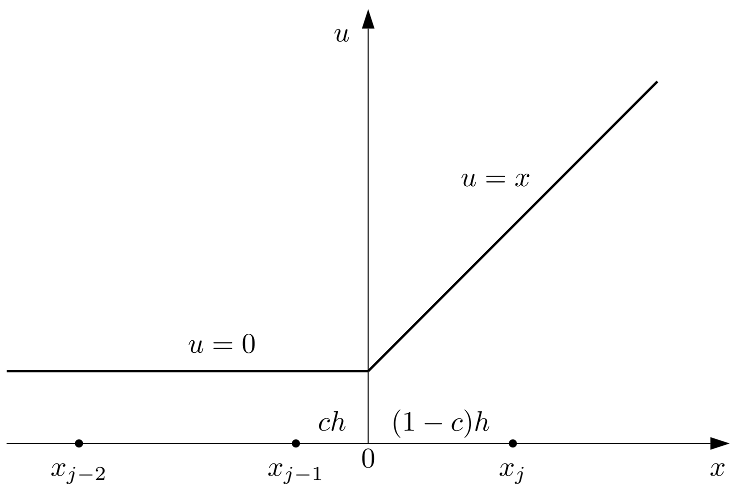


Figure 1: Non-smooth solution and a uniform grid.

## 2 Non-Smooth Solution

Consider the following non-smooth function (See Fig.1):

$$u = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{otherwise.} \end{cases} \tag{5}$$

This function can be an exact solution to the advection equation with

$$f = \begin{cases} 0 & \text{if } x \leq 0, \\ a & \text{otherwise.} \end{cases} \tag{6}$$

It does not satisfy the differential equation (1) at $x = 0$, but it does satisfy the integral form:

$$\int_C a\partial_x u \, dx = \int_C f \, dx, \tag{7}$$

where $C$ denotes a domain large enough to contain the point $x = 0$. Hence, it is an exact weak solution to the advection equation. This type of solution is typical in isentropic compression or expansion waves in compressible flows. Also, it is a local model for a non-differentiable point of a general non-smooth continuous solution.

We solve this problem numerically on a uniform grid with the boundary condition $u = 0$ at a far-left point ($x \ll 0$). In general, the non-differentiable point $x = 0$ can be anywhere in the grid. Let us assume that $x_{j-1} = -ch$ and $x_j = (1-c)h$, where $c \in [0, 1]$, so that the point $x = 0$ is located somewhere inside the interval $[x_{j-1}, x_j]$ (See Fig.1). Consider the difference scheme (2):

$$a\frac{3u_j - 4u_{j-1} + u_{j-2}}{2h} = f_j, \tag{8}$$

which can be used to find numerical solutions successively from the left boundary towards the right. The first non-trivial solution is obtained at the node $j$. Given the solution in $x \leq 0$, i.e., $u_{j-1} = u_{j-2} = 0$, we find that the scheme predicts the solution at $j$ as

$$u_j = \frac{2}{3} h, \tag{9}$$

where $f_j = a$ has been used since the node $j$ is located in the region $x > 0$. The exact solution at $x = x_j$ is given by

$$u_j^{exact} = (1 - c)h. \tag{10}$$

Therefore, the discretization error is given by

$$\left| u_j - u_j^{exact} \right| = \left| \frac{2}{3} h - (1 - c)h \right| = \left| c - \frac{1}{3} \right| h, \tag{11}$$

showing that the error vanishes if $c = 1/3$, and otherwise it is $O(h)$. Hence, the order of accuracy obtained in grid refinement is expected to be $O(h)$ in general. And the difference scheme produces the exact solution on a grid with $c = 1/3$.

The analysis can be generalized to higher-order difference schemes. An $n$-th higher-order upwind difference scheme can be expressed in the following form:

$$a \frac{\alpha u_j + \sum_{k=1}^{n} \alpha_k u_{j-k}}{h} = f_j, \tag{12}$$

where $\alpha$ and $\alpha_k$ are parameters defined by the scheme. At the node $j$ considered above, we have $u_{j-k} = 0$ for $k \geq 1$ and $f_j = a$ and therefore obtain

$$u_j = \frac{1}{\alpha} h. \tag{13}$$

The discretization error is given by

$$\left| u_j - u_j^{exact} \right| = \left| \frac{1}{\alpha} h - (1 - c)h \right| = \left| c - \frac{1}{\alpha} \right| h, \tag{14}$$

thus showing again that the error is generally $O(h)$ and vanishes if $c = 1/\alpha$.

The error committed at $j$ will propagate to the right, and therefore there is no hope that the numerical scheme will recover the design accuracy afterwards. Another observation is that the numerical error must be either zero or $O(h)$. This is because if the scheme eliminates the first-order error, there will be no more errors to eliminate; if it fails to eliminate the first-order error, the error must be $O(h)$.

## 3 Remark

It simply does not make sense to speak about the order of accuracy for non-smooth (non-differentiable) solutions.