Which is the Best Form for CFD: Differential, Integral, or Space-Time Form?

Hiroaki Nishikawa

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1 Finite-Difference, Finite-Volume, and Space-Time Schemes

Consider a conservation law in various forms:

$$\partial_t u + \partial_x f = 0, \qquad \text{Differential form,} \tag{1}$$

$$\frac{\partial}{\partial t}\int u\,dx + \oint f\,dx = 0, \qquad \text{Integral form,} \tag{2}$$

$$\oint \mathbf{F}\cdot d\mathbf{l} = 0, \qquad \text{Space-time integral form,} \tag{3}$$

where u is a solution variable, f is a flux, and

$$\mathbf{F} = (f, u), \quad d\mathbf{l} = (dx, dt). \tag{4}$$

Below, we discretize each form on a uniformly-spaced grid with spacings Δx in space and Δt in time, and compare the resulting discretizations. For simplicity, we only consider schemes of at least first-order accuracy, but require them to be conservative.

Finite-Difference:

We discretize the differential form as follows. Let u_j^n be a solution value at $x = x_j$ and at the time level n. We approximate the time derivative by the forward difference formula and the spatial derivative by the central finite-difference formula:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{f_{j+1/2}^n - f_{j-1/2}^n}{\Delta x} = 0, \tag{5}$$

where the spatial derivative has been approximated by the central difference formula based on the fluxes defined at halfway between two nodes to ensure discrete conservation. This is a typical conservative finite-difference scheme. The flux may be defined as

$$f_{j+1/2}^n = \frac{1}{2}(f_{j+1}^n + f_j^n) - a(u_{j+1}^n - u_j^n), \tag{6}$$

where a is a dissipation coefficient, and similarly for $f^n_{j-1/2}$.

Finite-Volume:

We discretize the integral form as follows. Let u^n_j be a cell-averaged solution value at a cell centered at $x = x_j$ and at the time level n. Consider the integral form applied to the cell. Divide the integral form by the cell volume Δx:

$$\frac{\partial}{\partial t}\left(\frac{1}{\Delta x}\int u\,dx\right) + \frac{1}{\Delta x}\oint f\,dx = 0, \tag{7}$$

then the first term on the left hand side is exactly the cell-average and the flux integral can be carried out exactly:

$$\frac{du_j}{dt} + \frac{1}{\Delta x}(f^n_{j+1/2} - f^n_{j-1/2}) = 0, \tag{8}$$

where the fluxes are defined at the cell boundaries, $j - 1/2$ and $j + 1/2$. Using the forward Euler time integration scheme, we obtain

$$\frac{u^{n+1}_j - u^n_j}{\Delta t} + \frac{f^n_{j+1/2} - f^n_{j-1/2}}{\Delta x} = 0. \tag{9}$$

The flux may be defined as

$$f^n_{j+1/2} = \frac{1}{2}(f^n_{j+1} + f^n_j) - a(u^n_{j+1} - u^n_j), \tag{10}$$

where a is a dissipation coefficient, and similarly for $f^n_{j-1/2}$.

Space-Time:

We discretize the space-time integral form as follows. Consider a control volume around $x = x_j$ with the left and right boundaries defined at $j - 1/2$ and $j + 1/2$, respectively, and the bottom and top boundaries defined at time levels n and $n + 1$, respectively. The space-time integral is evaluated by the midpoint rule at the bottom and top boundaries, and a one-point quadrature at time level n along the left and right boundaries, which is sufficient for first-order accuracy:

$$-f^n_{j-1/2}\Delta t - u^n_j\Delta x + f^n_{j+1/2}\Delta t + u^{n+1}_j\Delta x = 0, \tag{11}$$

which can be written as

$$\frac{u^{n+1}_j - u^n_j}{\Delta t} + \frac{f^n_{j+1/2} - f^n_{j-1/2}}{\Delta x} = 0, \tag{12}$$

where the fluxes can be evaluated as

$$f^n_{j+1/2} = \frac{1}{2}(f^n_{j+1} + f^n_j) - a(u^n_{j+1} - u^n_j), \tag{13}$$

where a is a dissipation coefficient, and similarly for $f^n_{j-1/2}$.

2 Final scheme

All schemes reduce to

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}(f_{j+1/2}^n - f_{j-1/2}^n) = 0, \tag{14}$$

with

$$f_{j+1/2}^n = \frac{1}{2}(f_{j+1}^n + f_j^n) - a(u_{j+1}^n - u_j^n). \tag{15}$$

Once implemented in a code, finite-difference, finite-volume, and space-time schemes are indistinguishable, except the initial solution. If the initial solution is set by cell-averages, then the scheme is finite-volume. If it is set by point values, then the scheme is finite-difference. Either way, the scheme may be space-time. But since the point values and cell averages differ by $O(\Delta x^2)$, all schemes are essentially the same. Which scheme works best? Well, whatever the initial solution is, numerical solutions evolve in exactly the same way. Therefore, the answer is that there is no single scheme that works better than others.

The same is true for second-order schemes. A second-order finite-difference scheme may be constructed by evaluating the spatial derivative at time level $n + 1/2$. Also, a second-order finite-volume scheme can be derived in the same way. A second-order space-time scheme can be derived by using the midpoint rule along boundaries. All schemes can be made identical.

So, if one claims that schemes derived from the space-time integral form are better than those derived from other forms, then others can claim the same because their schemes can also be derived from the space-time integral form with suitable techniques.

3 Remarks

1. It is not important what form of the governing equation the scheme has been derived from. As shown above, the same scheme can be derived from different forms. Note: It is almost always possible to derive the same scheme from any form of the governing equation by applying suitable numerical approximation techniques (e.g., quadrature, interpolation, etc.).

2. It is more meaningful to compare schemes in terms of discrete properties such as discrete conservation, energy/entropy-preserving, etc. Then, it makes sense to talk about which form of the governing equation is more convenient to work with to ensure such properties.

3. Differentiability is assumed in the differential or integral forms, and the space-time integral form has no such assumptions. But the resulting schemes are exactly the same. So, the differentiability assumption does not impact numerical schemes.

4. In higher dimensions, however, there may be differences. For example, the construction of space-time schemes in three dimensions require grids in four dimensions, which may lead to a unique scheme that cannot be easily reproduced from differential or integral forms.