From Diffusion Schemes to Naver-Stokes Schemes in Residual-Distribution Method

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Abstract

In this paper, we present a simple method to extend a diffusion scheme to the Navier-Stokes equations.

1 Navier-Stokes Equations

Consider the Navier-Stokes equations:

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{v}) = 0, \tag{1.1}$$

$$\partial_t(\rho \mathbf{v}) + \mathrm{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \mathrm{grad}\, p = \mathrm{div}\, \boldsymbol{\tau}, \tag{1.2}$$

$$\partial_t(\rho E) + \mathrm{div}(\rho \mathbf{v} H) = \mathrm{div}\,(\boldsymbol{\tau} \mathbf{v}) - \mathrm{div}\, \mathbf{q}, \tag{1.3}$$

where $\boldsymbol{\tau}$ is the viscous stress tensor and $\mathbf{q}$ is the heat flux vector. We write the system in the vector form:

$$\partial_t \mathbf{U} + \mathrm{div} \mathbf{F_i} = \mathrm{div} \mathbf{F_v}, \tag{1.4}$$

where $\mathbf{U}$ is the vector of conservative variables, $\mathbf{F_i}$ is the inviscid flux tensor, and $\mathbf{F_v}$ is the viscous flux tensor. Here, we consider constructing residual-distribution schemes of the form:

$$\mathbf{M}_j \frac{d\mathbf{U}_j}{dt} = \sum_{T \in \{T_j\}} \mathbf{\Phi}_j^T, \tag{1.5}$$

where $\{T_j\}$ is a set of triangles around node j, $\mathbf{M}_j$ is a mass matrix, $\mathbf{\Phi}_j^T$ is a partial cell-residual distributed to the node j within the triangle T which sum up to the cell-residual for the Navier-Stokes equations, $\mathbf{\Phi}^T$,

$$\mathbf{\Phi}^T = \sum_{k \in \{k_T\}} \mathbf{\Phi}_k^T, \tag{1.6}$$

where $\{k_T\} = \{1, 2, 3\}$ is a set of vertices of the triangle T. The cell-residual, $\mathbf{\Phi}^T$, is evaluated by the solution values and gradients that are assumed to be available at nodes (e.g., by gradient reconstruction or direct differentiation of a high-order element).

2 Residual-Distribution Schemes for Diffusion Equation

For the diffusion equation in two dimensions,

$$\partial_t u = \nu(\partial_{xx} u + \partial_{yy} u), \tag{2.1}$$

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we consider the diffusion schemes of the following form:

$$\left(\sum_{T \in \{T_j\}} \beta_j^T V_T \right) \frac{du_j}{dt} \;=\; \sum_{T \in \{T_j\}} \phi_j^T, \tag{2.2}$$

where V_T denotes the volume of the triangle T, ϕ_j^T is the partial residual distributed from the triangle T:

$$\phi_j^T = \beta_j^T \phi^T - \nu L^T \left\{ \nabla u^T - (\overline{p}_T, \overline{q}_T) \right\} \cdot \mathbf{m}_j^T, \tag{2.3}$$

and ϕ^T is the cell-residual for the diffusion equation, $\nu(u_{xx} + u_{yy})$:

$$\phi^T = \nu \left[(p_x)^T + (q_y)^T \right] V^T. \tag{2.4}$$

The gradient, ∇u^T, denotes the Green-Gauss gradient over the triangle. The vector of variables, (p, q), represents the solution gradient, and they are assumed to be available at nodes. The arithmetic average of the nodal gradient values are denoted by $(\overline{p}_T, \overline{q}_T)$. The derivatives of the gradients evaluated by the Green-Gauss formula are denoted by $(p_x)^T$ and $(q_y)^T$. We also assume that the vector $\mathbf{m}_i^T$ vanishes when summed over the triangle. Note that the second term in Equation (2.3) is the damping term [1]. The above diffusion scheme includes the LDA diffusion scheme Ref.[1]:

$$\beta_j^T = \frac{|\mathbf{n}_i|}{n_T}, \quad n_T = |\mathbf{n}_1| + |\mathbf{n}_2| + |\mathbf{n}_3|, \tag{2.5}$$

$$L^T = \frac{\alpha V_T}{L_r}, \tag{2.6}$$

$$\mathbf{m}_j^T = (D_{x_j}, -D_{y_j}), \tag{2.7}$$

$$D_{x_i} = \frac{\hat{n}_{y_k} - \hat{n}_{y_j}}{\hat{n}_{x_1}(\hat{n}_{y_2} - \hat{n}_{y_3}) + \hat{n}_{x_2}(\hat{n}_{y_3} - \hat{n}_{y_1}) + \hat{n}_{x_3}(\hat{n}_{y_1} - \hat{n}_{y_2})}, \tag{2.8}$$

$$D_{y_i} = \frac{\hat{n}_{x_k} - \hat{n}_{x_j}}{\hat{n}_{x_1}(\hat{n}_{y_2} - \hat{n}_{y_3}) + \hat{n}_{x_2}(\hat{n}_{y_3} - \hat{n}_{y_1}) + \hat{n}_{x_3}(\hat{n}_{y_1} - \hat{n}_{y_2})}, \tag{2.9}$$

and $(i, j, k) = (1, 2, 3)$, $(2, 3, 1)$, or $(3, 1, 2)$. Also, it includes the Lax-Wendroff diffusion scheme Ref.[1]:

$$\beta_j^T = \frac{1}{3}, \tag{2.10}$$

$$L^T = \frac{\alpha}{4}, \tag{2.11}$$

$$\mathbf{m}_j^T = \mathbf{n}_j^T. \tag{2.12}$$

$$\tag{2.13}$$

We may take $\alpha = 1$ as suggested in Ref.[1].

3 Extensions to Navier-Stokes Equations

Write the diffusion equation in the form:

$$u_t = \mathrm{div}\boldsymbol{\sigma}, \tag{3.1}$$

where $\boldsymbol{\sigma}$ is the diffusive flux defined by

$$\boldsymbol{\sigma} = (\nu u_x, \nu u_y). \tag{3.2}$$

Then, the diffusion scheme (2.2) can be expressed in terms of the diffusive flux as

$$\phi_j^T = \beta_j^T \phi^T - L^T \left\{ \boldsymbol{\sigma}^T - \overline{\boldsymbol{\sigma}}_T \right\} \cdot \mathbf{m}_j^T, \tag{3.3}$$

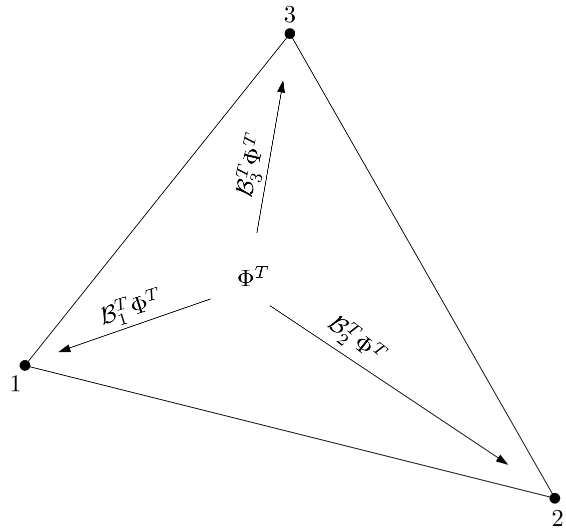

Figure 2.1: Distribution of a cell-residual to the set of vertices $\{i_T\} = \{1, 2, 3\}$. Each contribution is determined by multiplying the cell-residual by the distribution matrix, $\mathcal{B}_i^T$, where $i \in \{i_T\}$.

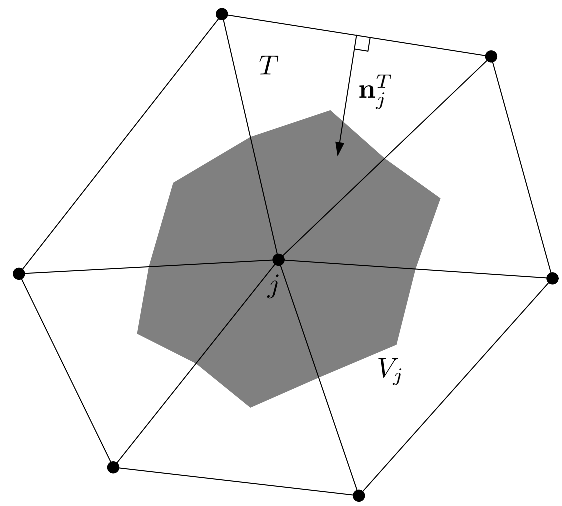

Figure 2.2: Median dual cell around a node j over the set of surrounding triangles, $\{T_j\}$. V_j is the dual cell area. $\mathbf{n}_j^T$ is the scaled inward normal (not drawn to scale) associated with a triangle $T \in \{T_j\}$.

where $\boldsymbol{\sigma}^T$ is the diffusive flux computed by the Green-Gauss formula over the triangle, $\overline{\boldsymbol{\sigma}}_T$ is the arithmetic average of the diffusion flux at nodes, and the cell-residual, ϕ^T, is the flux balance evaluated over the triangle by the Green-Gauss formula with the diffusive flux at nodes,

$$\phi^T = (\text{div}\boldsymbol{\sigma})^T V^T. \tag{3.4}$$

This is a useful way of looking at the diffusion scheme. That is, to extend the scheme to the viscous terms, we simply replace the diffusive flux by the viscous flux. For example, the viscous term in the momentum equations, div$\boldsymbol{\tau}$ is the viscous stress tensor, can be discretized as

$$(\boldsymbol{\phi}_v)_j^T = \beta_j^T (\boldsymbol{\phi}_v)^T - L^T \left\{ \boldsymbol{\tau}^T - \overline{\boldsymbol{\tau}}_T \right\} \mathbf{m}_j^T, \tag{3.5}$$

where

$$(\boldsymbol{\phi}_v)^T = (\text{div}\boldsymbol{\tau})^T V^T. \tag{3.6}$$

Here, $\boldsymbol{\tau}^T$ is the viscous stress tensor computed by the Green-Gauss formula over the triangle (just like the Galerkin discretization), $\overline{\boldsymbol{\tau}}_T$ is the arithmetic average of the viscous stresses at nodes (e.g., reconstructed), and $(\text{div}\boldsymbol{\tau})^T$ is the viscous flux divergence computed by the Green-Gauss formula over the triangle with the viscous stresses at nodes (e.g., reconstructed). The viscous term and the heat flux term in the energy equation can be discretized similarly. It thus leads to the following construction of a Navier-Stokes scheme:

$$\boldsymbol{\Phi}_j^T = \left(\frac{Re^T}{Re^T + 2} \mathcal{B}_j^T + \frac{2}{Re^T + 2} \mathcal{A}_j^T \right) \boldsymbol{\Phi}^T + \delta \boldsymbol{\Phi}_j^T, \tag{3.7}$$

where Re^T is a local Reynolds number defined over the triangle T, $\mathcal{B}_j^T$ is the distribution matrix for the inviscid part, $\mathcal{A}_j^T$ is the distribution matrix for the viscous part defined by the distribution coefficient of the diffusion scheme,

$$\mathcal{A}_j^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta_j^T & 0 & 0 \\ 0 & 0 & \beta_j^T & 0 \\ 0 & 0 & 0 & \beta_j^T \end{bmatrix} \tag{3.8}$$

in two dimensions, and $\delta\mathbf{\Phi}_j^T$ represents the damping terms which sum up to zero over the triangle T,

$$\delta\mathbf{\Phi}_j^T = \left[\begin{array}{c} 0 \\ -L^T \left\{ \boldsymbol{\tau}^T - \overline{\boldsymbol{\tau}}_T \right\} \mathbf{m}_j^T \\ -L^T \left\{ \boldsymbol{\tau}^T \overline{\mathbf{v}}^T - \overline{\boldsymbol{\tau}}_T \overline{\mathbf{v}}^T \right\} \cdot \mathbf{m}_j^T + L^T \left\{ \mathbf{q}^T - \overline{\mathbf{q}}_T \right\} \cdot \mathbf{m}_j^T \end{array} \right]. \tag{3.9}$$

[MY GUESS is that these damping terms play an important role in high-order schemes: the scheme may be inconsistent without these terms.] Note that the inviscid and viscous distribution matrices have been combined into one in Equation (3.7) in the way suggested in Ref.[2]. The damping term may be weighted similarly with the dissipation term from the inviscid part.

References

[1] Nishikawa, H., "Beyond Interface Gradient: A General Principle for Constructing Diffusion Schemes," *Proc. of 40th AIAA Fluid Dynamics Conference and Exhibit*, AIAA Paper 2010-5093, Chicago, 2010.

[2] Nishikawa, H., "A First-Order System Approach for Diffusion Equation. II: Unification of Advection and Diffusion," *J. Comput. Phys.*, Vol. 229, 2010, pp. 3989–4016.