1. Comparison of Discretization Methods for Tetrahedral Grids

for the same number of Total Discrete Unknowns (TDU).

	DpC	EG-nodes	Tets	TDU	Storage	Inv	Visc	Grad	Stencil	Loop	Qpts	Aux Comp	Mesh
HNS-CCFV(1st)	20	0.25M	1.5M	30M	30M	2nd	2nd	1st	1	face=12M	12M	N/A	Linear
CC-FV(2nd)	5	1M	6M	30M	120M	2nd	2nd	1st	2	face=12M	12M	LLSQ(3)	Linear
P1-DG(BR2)	20	0.25M	1.5M	30M	84M	2nd	2nd	1st	1	face=3M	10M	Stress/Heat	Curved
P1-CG/RD	5	6M	36M	30M	30M	2nd	2nd	1st	1	elm=36M	144M	N/A	Linear
EB(2nd)	5	6M	36M	30M	120M	2nd	2nd	1st	2	edge=42M	42M	LLSQ(3)	Linear
HNS-EB(2nd)	20	1.5M	9M	30M	120M	**3rd**	2nd	2nd	2	edge=10M	10M	LLSQ(3)	Linear
CC-FV(3rd)	5	1M	6M	30M	300M	**3rd**	**3rd**	2nd	2	face=12M	36M	QLSQ(9)	Curved
HNS-EB(3rd)	20	1.5M	9M	30M	120M	**3rd**	**3rd**	**3rd**	3	edge=10M	10M	QLSQ(3)	Linear
rDG(P1P2)	20	0.25M	1.5M	30M	210M	**3rd**	**3rd**	2nd	2	face=3M	18M	LLSQ(3)	Curved
P2-DG(BR2)	50	0.1M	0.6M	30M	84M	**3rd**	**3rd**	2nd	1	face=1.2M	10M	Stress/Heat	Curved
P2-CG/RD	5	0.75M	4.5M	30M	30M	**3rd**	**3rd**	2nd	2	elm=5M	45M	N/A	Curved

Tetrahedral grid: Tetra ~ 6*nodes, Edges ~ 7*nodes, Faces ~ 12*nodes, P2-element-nodes ~ 8*nodes

DpC: Discrete unknowns per cell/node for 3D Navier-Stokes (5 variables).

EG-nodes: Equivalent-Grid nodes. Nodes of tetra grids to yield the same 30M discrete unknowns.

Tets: The number of tetrahedra. **TDU:** Total Discrete Unknowns

Storage: Solution values and gradients/viscous-stresses/heat-fluxes.

Inv/Visc/Grad: Inviscid/Viscous/Gradient accuracy order **Stencil:** Extent of the residual stencil. 1=neighbors only.

Loop: over which the residual is computed. **Qpts:** Flux evaluations.

Aux Comp: Auxiliary computations. L/QLSQ(n) = Linear/Quadratic LSQ gradients (n=3), with Hessian (n=9).

HNS-EB: Edge-based method with the **HNS20 system**. **P1/P2-DG**: Modal basis is assumed.

Remarks:
- All lead to the same size of a global system of residuals: 30M residuals/unknowns.
- Grid size varies to match the total number of discrete unknowns. See EG-nodes.
- 'Expensive' methods are not really expensive (if required accuracy is obtained on such coarse grids).
- No intention to single out the "best" method. You choose one that fits your needs and available resource.

Hiroaki Nishikawa, 04/24/2017

2. Comparison of Discretization Methods for Tetrahedral Grids

for a given grid: You have a grid, and wonder which method to use.

	DpC	GG-nodes	Tets	TDU	Storage	Inv	Visc	Grad	Stencil	Loop	Qpts	Aux Comp	Mesh
HNS-CCFV(1st)	20	6M	36M	720M	720M	2nd	2nd	1st	1	face=72M	72M	N/A	Linear
CC-FV(2nd)	5	6M	36M	180M	720M	2nd	2nd	1st	2	face=72M	72M	LLSQ(3)	Linear
P1-DG(BR2)	20	6M	36M	180M	2016M	2nd	2nd	1st	1	face=72M	252M	Stress/Heat	Curved
P1-CG/RD	5	6M	36M	30M	30M	2nd	2nd	1st	1	elm=36M	144M	N/A	Linear
EB(2nd)	5	6M	36M	30M	120M	2nd	2nd	1st	2	edge=42M	42M	LLSQ(3)	Linear
HNS-EB(2nd)	20	6M	36M	120M	480M	**3rd**	2nd	2nd	2	edge=42M	42M	LLSQ(3)	Linear
CC-FV(3rd)	5	6M	36M	180M	1800M	**3rd**	**3rd**	2nd	2	face=72M	216M	QLSQ(9)	Curved
HNS-EB(3rd)	20	6M	36M	120M	480M	**3rd**	**3rd**	**3rd**	3	edge=42M	42M	QLSQ(3)	Linear
rDG(P1P2)	20	6M	36M	720M	5040M	**3rd**	**3rd**	2nd	2	face=72M	432M	LLSQ(3)	Curved
P2-DG(BR2)	50	6M	36M	1800M	5040M	**3rd**	**3rd**	2nd	1	face=72M	612M	Stress/Heat	Curved
P2-CG/RD	5	6M	36M	240M	240M	**3rd**	**3rd**	2nd	2	elm=36M	360M	N/A	Curved

Tetrahedral grid: Tetra ~ 6*nodes, Edges ~ 7*nodes, Faces ~ 12*nodes, P2-element-nodes ~ 8*nodes

DpC: Discrete unknowns per cell/node for 3D Navier-Stokes (5 variables).
GG-nodes: Given-Grid nodes. Nodes of a given tetrahedral grid. 6M is an example.
Tets: The number of tetrahedra. **TDU:** Total Discrete Unknowns
Storage: Solution values and gradients/viscous-stresses/heat-fluxes.
Inv/Visc/Grad: Inviscid/Viscous/Gradient accuracy order **Stencil:** Extent of the residual stencil. 1=neighbors only.
Loop: over which the residual is computed. **Qpts:** Flux evaluations.
Aux Comp: Auxiliary computations. L/QLSQ(n) = Linear/Quadratic LSQ gradients (n=3), with Hessian (n=9).
HNS-EB: Edge-based method with the **HNS20 system**. **P1/P2-DG**: Modal basis is assumed.

Remarks:
- Grid is given (linear or curved), which has 36M tetrahedra and 6M nodes.
- Usually, 3rd-order methods are always more expensive than 2nd-order methods on the same grid.
- Some methods may look really expensive, but provide higher-order accuracy or higher resolution (more TDU).
- No intention to single out the “best” method. You choose one that fits your needs and available resource.

Hiroaki Nishikawa, 04/24/2017

3. Comparison of Discretization Methods for Tetrahedral Grids

implicit solver construction for the same number of Total Discrete Unknowns (TDU).

	DpC	EG-nodes	Tets	TDU	Res extent	Loop	Jack Block	Compact Nghbrs	Compact Off-Diag	Jac/Res order	Cond. #
HNS-CCFV(1st)	20	0.25M	1.5M	30M	1	face=12M	20x20	4	2.4G	1st/1st	$O(h^{-1})$
CC-FV(2nd)	5	1M	6M	30M	2	face=12M	5x5	4	0.6G	1st/2nd	$O(h^{-2})$
P1-DG(BR2)	20	0.25M	1.5M	30M	1	face=3M	20x20	4	2.4G	2nd/2nd	$O(h^{-2})$
P1-CG/RD	5	6M	36M	30M	1	elm=36M	5x5	14	2.1G	2nd/2nd	$O(h^{-2})$
EB(2nd)	5	6M	36M	30M	2	edge=42M	5x5	14	2.1G	1st/2nd	$O(h^{-2})$
HNS-EB(2nd)	20	1.5M	9M	30M	2	edge=10M	20x20	14	8.4G	1st/2nd	$O(h^{-1})$
CC-FV(3rd)	5	1M	6M	30M	2	face=12M	5x5	4	0.6G	1st/3rd	$O(h^{-2})$
HNS-EB(3rd)	20	1.5M	9M	30M	3	edge=10M	20x20	14	8.4G	1st/2nd	$O(h^{-1})$
rDG(P1P2)	20	0.25M	1.5M	30M	2	face=3M	20x20	4	2.4G	2nd/3rd	$O(h^{-2})$
P2-DG(BR2)	50	0.1M	0.6M	30M	1	face=1.2M	50x50	4	6.0G	3rd/3rd	$O(h^{-2})$
P2-CG/RD	5	0.75M	4.5M	30M	2	elm=5M	5x5	64	1.2G	3rd/3rd	$O(h^{-2})$

Tetrahedral grid: Tetra ~ 6*nodes, Edges ~ 7*nodes, Faces ~ 12*nodes, P2-element-nodes ~ 8*nodes

DpC: Discrete unknowns per cell/node for 3D Navier-Stokes (5 variables).
EG-nodes: Equivalent-Grid nodes. Nodes of tetra grids to yield the same 30M discrete unknowns.
Tets: The number of tetrahedra. **TDU:** Total Discrete Unknowns
Res extent: Extent of the residual stencil. 1=neighbors only.
Loop: over which the residual/Jacobian is computed. **Compact Nghbrs:** Number of nghbrs for Jacobian.
Jac Block: Jacobian block size. **Compact Off-Diag:** Total off-diagonal elements for a compact stencil.
Jac/Res order: Order of approximation of compact-Jacobian and residual. **Cons. #:** Condition number order of Jac.
HNS-EB: Edge-based method with the **HNS20 system**

Remarks:
- All lead to the same size of a global system of residuals: 30M residuals/unknowns and 30Mx30M Jacobian.
- Grid size varies to match the total number of discrete unknowns.
- JFNK solver is applicable to all; a non-exact Jac is a good candidate for a preconditioner matrix.
- Cost of a solver is likely to depend on the number of total residual evaluations towards convergence.
- Assumption is that required accuracy is obtained on each EG.

Hiroaki Nishikawa, 04/24/2017

Discretization Methods

Those used to generate the tables. There are numerous variants.

HNS-CCFV(1st)	1st-order cell-centered FV discretization with 2nd-order inviscid terms with the use of gradient variables. It is known this scheme can achieve 2nd-order accuracy for inviscid/viscous terms and 1st-order for gradients.
CC-FV(2nd)	Linear reconstruction with linear LSQ gradients. Viscous scheme with a damping term.
P1-DG(BR2)	Linear modal DG with the BR2 viscous method. P1 polynomials for viscous stresses and heat fluxes. 3 and 1 quadrature points for surface and volume integrals (tetrahedra).
P1-CG/RD	P1 continuous Galerkin (SUPG). Residual-distribution method with a SUPG distribution matrix. Quadrature points are based on flux/stabilization evaluations per element: 4 for a P1 tetrahedra.
EB(2nd)	Edge-based inviscid discretization with an edge-based implementation of P1-CG viscous discretization.
HNS-EB(2nd)	2nd-order edge-based discretization with a hyperbolic viscous formulation. 3rd-order for inviscid terms and 2nd-order gradients.
CC-FV(3rd)	Quadratic reconstruction with quadratic LSQ gradients/Hessian using neighbors and their neighbors. Viscous scheme with a damping term. 3 quadrature points for surface (tetrahedra).
HNS-EB(3rd)	Third-order edge-based discretization with a hyperbolic viscous formulation. 3rd-order for inviscid/viscous terms and gradients. Quadratic LSQ (with neighbors and their neighbors), but only gradients are needed.
rDG(P1P2)	Linear modal DG with a reconstructed quadratic flux (by linear LSQ applied to the gradients) and the BR2 viscous method. 4 and 4 quadrature points for surface and volume integrals (tetrahedra).
P2-DG(BR2)	Quadratic modal DG with the BR2 viscous method. P2 polynomials for viscous stresses and heat fluxes. 6 and 5 quadrature points for surface and volume integrals (tetrahedra).
P2-CG	P2 continuous Galerkin (SUPG). Residual-distribution method with a SUPG distribution matrix. Quadrature points are based on flux/stabilization evaluations per element: 10 for a P2 tetrahedra.

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Remarks

- Tables indicate that no universal winner exists. Each has pros and cons.
- Tables may help one choose that fits your needs and available resource, or researchers/developers identify areas for improvements, or just serve as a starting point for further discussion.
- Each method has numerous variants. Numbers are likely to change with the variation.
- Use of other types of elements (e.g., prism, hex) will change the numbers also.
- Stability is another important consideration.
- There exist many other 2nd/3rd-order methods.
- Grid adaptation is a key technology for all methods.
- There are just so many factors to be considered for a true comparison. These tables are an attempt to provide some quantitative comparison among representative methods.