# Three-Dimensional Stream Functions

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## 1 Stream Functions in Three Dimensions

Consider the continuity equation for incompressible flows in three dimensions,

$$\mathrm{div}\vec{V} = 0 \tag{1}$$

where $\vec{V}$ is the velocity vector. Owing to the identity $\mathrm{div}(\mathrm{curl}\vec{a}) = 0$, the velocity defined by

$$\vec{V} = \mathrm{curl}\vec{A}, \tag{2}$$

where $\vec{A}$ is the so-called vector potential, satisfies the continuity equation identically. Now suppose that we set

$$\vec{A} = \psi\, \mathrm{grad}\chi \tag{3}$$

where $\psi$ and $\chi$ are scalar functions. Then,

$$\begin{aligned} \vec{V} &= \mathrm{curl}(\psi\, \mathrm{grad}\chi) &(4)\\ &= \psi\, \mathrm{curl}(\mathrm{grad}\chi) + \mathrm{grad}\psi \times \mathrm{grad}\chi. &(5) \end{aligned}$$

The first term vanishes identically, and we have

$$\vec{V} = \mathrm{grad}\psi \times \mathrm{grad}\chi. \tag{6}$$

This shows that the velocity vector is orthogonal to each gradient vector, that is, tangent to the surfaces defined by $\psi$ = constant and $\chi$ = constant. Therefore the functions $\psi$ and $\chi$ are stream functions in three dimensions (See Figure 1).

The stream functions thus obtained are associated with the volume flow. Consider a volume flow $Q$ through a section which is bounded by four stream surfaces (See Figure 2) given by

$$Q = \iint_S \vec{V} \cdot d\vec{S}. \tag{7}$$

By Stokes' theorem, this reduces to the line integral,

$$Q = \oint_{\partial S} \psi\, \mathrm{grad}\chi \cdot d\vec{l}. \tag{8}$$

Note that on each boundary segment, one of the functions is constant. Hence we have

$$Q = \psi_2 \int_{AB} \mathrm{grad}\chi \cdot d\vec{l} + \psi_1 \int_{CD} \mathrm{grad}\chi \cdot d\vec{l}. \tag{9}$$

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By the definition, $d\psi = \text{grad}\psi \cdot d\vec{l}$, we obtain

$$Q = (\psi_1 - \psi_2)(\chi_1 - \chi_2). \tag{10}$$

Therefore the volume flow is given by the product of the differences of the stream functions.

It is well known that the stream function satisfies a Poisson equation in two dimensions. To obtain the equation in three dimensions, we first substitute (2) into the definition of the vorticity ($\vec{\omega} = \text{curl}\vec{V}$) to get

$$\text{curl}\,\text{curl}\vec{A} = \vec{\omega}. \tag{11}$$

Using the identity, $\text{grad}\,(\text{div}\vec{A}) - \text{curl}\,\text{curl}\vec{A} = \text{div}\,\text{grad}\vec{A}$, we obtain

$$-\mathbf{div}\,\mathbf{grad}\vec{A} + \text{grad}\,(\text{div}\vec{A}) = \vec{\omega}. \tag{12}$$

In terms of the stream functions, this is

$$-\mathbf{div}\,\mathbf{grad}(\psi\,\text{grad}\chi) + \text{grad}\,\text{div}(\psi\,\text{grad}\chi) = \vec{\omega}. \tag{13}$$

It can be written, as in [1], also in the following form,

$$\mathbf{L}(\psi)\,\text{grad}\chi - \mathbf{L}(\chi)\,\text{grad}\psi = \vec{\omega} \tag{14}$$

where the operator $\mathbf{L}$ is defined by[1]

$$\mathbf{L}(\psi) = -\mathbf{I}\,\text{div}(\text{grad}\psi) + \mathbf{grad}(\text{grad}\psi). \tag{15}$$

This is the governing equation of the stream functions in three dimensions.

# 2 Special Cases

## 2.1 Two Dimensional Flows

In two dimension with coordinates $(x, y)$, we set $\chi = z$ and $\psi = \psi(x, y)$, i.e. we choose the planes perpendicular to z-axis to be stream surfaces. From (6), the velocity is given by

$$\vec{V} = \text{grad}\psi \times \vec{e_z} \tag{16}$$

where $\vec{e_z}$ is the unit vector in z-axis. The velocity components $(u, v)$ are therefore given by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{17}$$

From (10), we find that the volume flow between two stream lines is given by

$$Q = \psi_1 - \psi_2 \tag{18}$$

where we have set $\chi_1 - \chi_2 = 1$. The equation that $\psi$ satisfies is obtained by (14). With $\chi = z$ and $\psi = \psi(x, y)$, (14) simplifies to

$$-\text{div}(\text{grad}\psi) = \omega_z. \tag{19}$$

[1] $\mathbf{grad}$ is defined by $\vec{a} = (\mathbf{grad}\vec{a})\,d\vec{l}$ for an arbitrary vector $\vec{a}$.

### 2.2 Axisymmetric Flows

We take the spherical coordinate system $(r, \theta, \phi)$, and assume that the flow is independent of $\phi$, i.e. axisymmetric. We then take $\chi = \phi$ and $\psi = \psi(r, \theta)$ so that $\phi =$ constant constitues a family of stream surfaces. In this case, from (6), the velocity is given by

$$\vec{V} = \mathrm{grad}\psi \times \frac{1}{r\sin\theta}\,\vec{e_\phi} \tag{20}$$

where $\vec{e_\phi}$ is the unit vector in $\phi$ direction. Its components, $(u_r, u_\theta)$, are therefore given by

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{21}$$

The volume flow is obtained again by the difference of the stream function. The equation that $\psi$ satisfies is, again from (14),

$$-\frac{1}{r\sin\theta}\,\mathrm{div}(\mathrm{grad}\psi) = \omega_\phi. \tag{22}$$

Similar results can be obtained for the cylindrical coordinate system.

## 3 Irrotational Flows

In two-dimensional or axisymmetric irrotational flows, it is clear that the stream function $\psi$ satisfies Laplace equation. Therefore $\psi$ can be determined, up to an arbitrary constant, by solving Laplace equation for which various techniques are available. In three dimensions, the stream functions satisfy

$$\mathbf{L}(\psi)\,\mathrm{grad}\chi - \mathbf{L}(\chi)\,\mathrm{grad}\psi = \mathbf{0}. \tag{23}$$

Note that this is a nonlinear system of equations, and that we have three equations for two unknowns.

## References

[1] Pozrikidis, C., *Introduction to Theoretical and Computational Fluid Dynamics*, Oxford University Press, 1997.