

Accuracy problem and a possible solution are described in:

On High-Order Fluctuation-Splitting Schemes for Navier-Stokes Equations, Computational Fluid Dynamics 2004: Proceedings of the Third International Conference on Computational Fluid Dynamics, ICCFD, Toronto, 12-16 July 2004, Springer, 2006.

# On the Accuracy of Fluctuation-Splitting Schemes for Advection-Diffusion Equation

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## 1 2nd-Order LDA-Galerkin Scheme

We consider high-order spatial discretization of the following model equation over a triangular grid.

$$\partial_t u + au_x + bu_y = \mu \operatorname{div}(\operatorname{grad}u). \quad (1)$$

## 2 Truncation Error for Viscous Part

### 2.1 Standard Galerkin (STD Galerkin)

The standard Galerkin

$$S_j \frac{du_j}{dt} = -\frac{\mu}{2} \sum_{T \in \{T_j\}} (\operatorname{grad}u)^T \cdot \mathbf{n}_T \quad (2)$$

where  $S_j$  is the area of the median dual cell and  $\mathbf{n}_T$  is the scaled inward normal for the triangle  $T$  of the edge opposite to the node  $j$ .

On a regular triangular grid, we expand a solution and insert it into the update formula to find the truncation error. At convergence, we have

$$\begin{aligned} 0 &= (u_{xx} + u_{yy}) + \frac{h^2}{12} [u_{xxxx} + u_{yyyy}] \\ &\quad + \frac{h^4}{360} [u_{xxxxx} + u_{yyyyy}]. \end{aligned} \quad (3)$$

The error is  $\mathcal{O}(h^2)$ . Also, note that the error depends on the 4th (and higher) derivatives of the solution, meaning the scheme is exact for polynomial solutions of degree 3.

## 2.2 High-Order Discretization: Gradient Reconstruction (GR)

$$u_j^{n+1} = u_j^n - \omega \frac{\mu}{2} \sum_{T \in \{T_j\}} (\text{gradu})_{high}^T \cdot \mathbf{n}_T \quad (4)$$

On a regular grid, the resulting scheme has the following discretization error.

$$\begin{aligned} 0 &= (\Delta u) + \frac{h^2}{12} [\partial_{xx} + \partial_{xy} + \partial_{yy}] (\Delta u) \\ &+ h^4 \left[ \frac{1}{360} \partial_{xxxx} + \frac{1}{48} \partial_{xxxxy} + \frac{43}{720} \partial_{xxxyy} + \frac{1}{48} \partial_{xyyyy} + \frac{1}{360} \partial_{yyyyy} \right] (\Delta u) \\ &+ \frac{h^4}{24} u_{xxxxyyy} \end{aligned} \quad (5)$$

where  $\Delta u = u_{xx} + u_{yy}$ .

## 2.3 Gradient-Based Discretization (GBD) of Laplacian

Defining the viscous fluctuation as

$$\phi_v^T = \mu \sum_{i \in \{i_T\}} \frac{1}{2} (\text{gradu})_i \cdot \mathbf{n}_i, \quad (6)$$

we distribute it with the equal weight 1/3,

$$S_j u_j^{n+1} = S_j u_j^n - \sum_{T \in \{T_j\}} \frac{1}{3} \phi_v^T \quad (7)$$

or equivalently,

$$S_j u_j^{n+1} = S_j u_j^n - \mu \sum_{T \in \{T_j\}} (\bar{u}_{xT}, \bar{u}_{yT}) \cdot \mathbf{n}_T \quad (8)$$

On a regular grid, inserting the expanded solution into the update, we obtain

$$\begin{aligned} 0 &= (\Delta u) + \frac{h^2}{3} [\partial_{xx} + \partial_{xy} + \partial_{yy}] (\Delta u) \\ &+ h^4 \left[ \frac{2}{45} \partial_{xxxx} + \frac{1}{12} \partial_{xxxxy} + \frac{3}{20} \partial_{xxxyy} + \frac{1}{12} \partial_{xyyyy} + \frac{2}{45} \partial_{yyyyy} \right] (\Delta u) \\ &+ \frac{h^4}{18} u_{xxxxyyy}. \end{aligned} \quad (9)$$

## 2.4 SUPG for Laplacian

Viscous fluctuation is the same as in the previous scheme, but distributed by the SUPG scheme whose distribution coefficient is given by

$$\beta = \frac{1}{3} \left( 1 + \tau \frac{k_i}{\sum \max(0, k_i)} \right). \quad (10)$$

Note that there is no one-target distribution.

$$\begin{aligned} 0 &= (\Delta u) - \frac{\tau h}{3a} [a\partial_x + b\partial_y] (\Delta u) \\ &+ \frac{h^2}{3} (\partial_{xx} + \partial_{xy} + \partial_{yy}) \Delta u + \mathcal{O}(h^3) \end{aligned} \quad (11)$$

## 2.5 LDA for Laplacian

Viscous fluctuation is the same as in the previous scheme, but distributed by the LDA scheme.

$$\begin{aligned} 0 &= (\Delta u) - \frac{h}{2a} [a\partial_x + b\partial_y] (\Delta u) \\ &+ \frac{h^2}{12a} [2a(2\partial_{xx} + \partial_{xy} + 2\partial_{yy}) \Delta u + 3au_{xyyy} - au_{xxyy} + 3b(u_{xxyy} + u_{xxxy})] \\ &+ \mathcal{O}(h^3) \end{aligned} \quad (12)$$

## 3 Truncation Error for Advective Part

In this section, "2nd-order" and "3rd-order" refer to the order of accuracy of the fluctuation (2nd-order fluctuation or 3rd-order fluctuation).

### 3.1 2nd-Order LDA

$$\begin{aligned} 0 &= -(au_x + bu_y) + \frac{h}{2a} [a\partial_x + b\partial_y] (au_x + bu_y) \\ &- \frac{h^2}{6} (au_{xxx} + bu_{yyy} + 3bu_{xyy} + 3bu_{xxy}) + \mathcal{O}(h^3) \end{aligned} \quad (13)$$

### 3.2 3rd-Order LDA

$$\begin{aligned} 0 &= -(au_x + bu_y) + \frac{h}{2a} [a\partial_x + b\partial_y] (au_x + bu_y) \\ &- \frac{h^2}{12a} [2a\partial_{xx} + (a + 2b)\partial_{xy} + (a + b)\partial_{yy}] (au_x + bu_y) \\ &+ \frac{h^3}{24a} [a\partial_{xxx} + b\partial_{yyy} + (a + b)\partial_{xxy} + (a + b)\partial_{xyy}] (au_x + bu_y) + \mathcal{O}(h^4) \end{aligned}$$

This is not a practical scheme itself, but will be partly used in effect when we combine an advection and a diffusion schemes.

### 3.3 2nd-Order 1/3

$$0 = -(au_x + bu_y) - \frac{h^2}{6} (\partial_{xx} + \partial_{xy} + \partial_{yy}) (au_x + bu_y) + \mathcal{O}(h^4) \quad (14)$$

### 3.4 3rd-Order 1/3

$$0 = -(au_x + bu_y) - \frac{5h^2}{36} (\partial_{xx} + \partial_{xy} + \partial_{yy}) (au_x + bu_y) + \mathcal{O}(h^4) \quad (15)$$

### 3.5 2nd-Order SUPG

$$\begin{aligned} 0 = & -(au_x + bu_y) + \frac{\tau h}{3a} (a\partial_x + b\partial_y) (au_x + bu_y) \\ & - \frac{h^2}{6} (\partial_{xx} + \partial_{xy} + \partial_{yy}) (au_x + bu_y) \\ & + \frac{\tau h^3}{36a} (6ab u_{xxyy} + 4ab u_{xxxy} + a^2 u_{xxxx} + 4ab u_{xyyy} + b^2 u_{yyyy}) \\ & + \mathcal{O}(h^4) \end{aligned} \quad (16)$$

### 3.6 3rd-Order SUPG

$$\begin{aligned} 0 = & -(au_x + bu_y) + \frac{\tau h}{3a} (a\partial_x + b\partial_y) (au_x + bu_y) \\ & - \frac{5h^2}{36} (\partial_{xx} + \partial_{xy} + \partial_{yy}) (au_x + bu_y) \\ & + \frac{\tau h^3}{36a} [a\partial_{xxx} + b\partial_{yyy} + (a+b)\partial_{xxy} + (a+b)\partial_{xyy}] (au_x + bu_y) \\ & + \mathcal{O}(h^4) \end{aligned} \quad (17)$$

## 4 Truncation Error for Advection-Diffusion Equation

If we combine a viscous scheme and an advection scheme as a simple sum, the truncation error is also a simple sum. Then we immediately find that the resulting scheme is not as accurate as the separate ones.

For example, the sum of the 2nd-order schemes (Galerkin + LDA) gives

$$\begin{aligned}
0 &= [-(au_x + bu_y) + \nu \Delta u] + \frac{h}{2a} [a\partial_x + b\partial_y] (au_x + bu_y) \\
&\quad - \frac{h^2}{6} (au_{xxx} + bu_{yyy} + 3bu_{xyy} + 3bu_{xxy}) + \nu \frac{h^2}{12} [u_{xxxx} + u_{yyyy}] \\
&\quad + \mathcal{O}(h^3)
\end{aligned} \tag{18}$$

If we use 3rd-order GR scheme for diffusion and the 3rd-order LDA for the advective part as a simple sum, we obtain

$$\begin{aligned}
0 &= [-(au_x + bu_y) + \nu \Delta u] + \frac{h}{2a} [a\partial_x + b\partial_y] (au_x + bu_y) \\
&\quad - \frac{h^2}{12a} [2a\partial_{xx} + (a+2b)\partial_{xy} + (a+b)\partial_{yy}] (au_x + bu_y) + \nu \frac{h^2}{12} [\partial_{xx} + \partial_{xy} + \partial_{yy}] (\Delta u) \\
&\quad + \frac{h^3}{24a} [a\partial_{xxx} + b\partial_{yyy} + (a+b)\partial_{xxy} + (a+b)\partial_{xyy}] (au_x + bu_y) \\
&\quad + \nu h^4 \left[ \frac{1}{360} \partial_{xxxx} + \frac{1}{48} \partial_{xxxy} + \frac{43}{720} \partial_{xxyy} + \frac{1}{48} \partial_{xyyy} + \frac{1}{360} \partial_{yyyy} \right] (\Delta u) \\
&\quad + \nu \frac{h^4}{24} u_{xxxyyy} + \mathcal{O}(h^4)
\end{aligned}$$

Alternative approach is to distribute the sum of the fluctuation by a combined distribution coefficient. Combination of the 3rd order schemes (the fluctuation is consistent with the quadratic element).

$$S_j u_j^{n+1} = S_j u_j^n - \sum_{T \in \{T_j\}} (\beta_{LDA} + \nu/3) (\phi^T + \phi_v^T) \tag{19}$$

$$\begin{aligned}
0 &= (1 + \nu) [-(au_x + bu_y) + \nu \Delta u] \\
&\quad + \frac{h}{2a} (a\partial_x + b\partial_y) [-(au_x + bu_y) + \nu \Delta u] + \mathcal{O}(h^2)
\end{aligned}$$

SUPG can also be used.

$$S_j u_j^{n+1} = S_j u_j^n - \sum_{T \in \{T_j\}} \frac{1}{3} \left( 1 + \tau \frac{k_i}{\sum \max(0, k_i)} \right) (\phi^T + \phi_v^T) \tag{20}$$

Using 3rd-order fluctuation for the advective part, we get

$$\begin{aligned}
0 &= [-(au_x + bu_y) + \nu \Delta u] \\
&\quad - \frac{\tau h}{3a} (a\partial_x + b\partial_y) [-(au_x + bu_y) + \nu \Delta u] \\
&\quad - \frac{5h^2}{36} (\partial_{xx} + \partial_{xy} + \partial_{yy}) (au_x + bu_y) + \nu \frac{h^2}{3} (\partial_{xx} + \partial_{xy} + \partial_{yy}) \Delta u \\
&\quad + \mathcal{O}(h^3).
\end{aligned} \tag{21}$$

Using the 2nd-order fluctuation, we get

$$\begin{aligned} 0 &= [-(au_x + bu_y) + \nu \Delta u] \\ &\quad - \frac{\tau h}{3a} (a\partial_x + b\partial_y) [-(au_x + bu_y) + \nu \Delta u] \\ &\quad - \frac{h^2}{6} (\partial_{xx} + \partial_{xy} + \partial_{yy}) (au_x + bu_y) + \nu \frac{h^2}{3} (\partial_{xx} + \partial_{xy} + \partial_{yy}) \Delta u \\ &\quad + \mathcal{O}(h^3). \end{aligned} \tag{22}$$