Area Change Formula

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1 Area Change Formula

Consider a triangular element T that deformes due to the movement of its vertices $\delta\vec{x}_i = [\delta x_i, \delta y_i]^t$ where $i = 1, 2, 3$. The new vertex coordinates are given by

$$\vec{x}_i' = \vec{x}_i + \delta\vec{x}_i, \tag{1}$$

and these form a new triangle T'. The change in the area is

$$\begin{aligned} \Delta S_T &= S_{T'} - S_T \\ &= \frac{1}{2}\sum_i x'_i \Delta y'_i - \frac{1}{2}\sum_i x_i \Delta y_i. \end{aligned} \tag{2}$$

Substitute (1) to get

$$\begin{aligned} \Delta S_T &= \frac{1}{2}\sum_i (x_i + \delta x_i)\Delta(y_i + \delta y_i) - \frac{1}{2}\sum_i x_i \Delta y_i \\ &= \frac{1}{2}\sum_i \left[x_i \Delta(\delta y_i) + \delta x_i \Delta y_i + \delta x_i \Delta(\delta y_i)\right] \end{aligned} \tag{3}$$

in which the third term in the square bracket may be neglected for small deformation, yielding

$$\Delta S_T = \frac{1}{2}\sum_i \left[-\delta y_i \Delta x_i + \delta x_i \Delta y_i\right] = \frac{1}{2}\sum_i \delta\vec{x}_i \cdot \vec{n}_i \tag{4}$$

where $\vec{n}_i = [\Delta y_i, -\Delta x_i]^t$ denotes the scaled inward normal vector opposite to the vertex i. Introducing deformation speed, $\vec{v}_i = \dfrac{\delta\vec{x}_i}{dt}$ we finally obtain

$$\frac{dS_T}{dt} = \lim_{\Delta t \to 0} \frac{\Delta S_T}{\Delta t} = \frac{1}{2}\sum_i \vec{v}_i \cdot \vec{n}_i \tag{5}$$

which is the formula for the rate of change of area per unit time due to the movement of the vertices [*].

2 Solution Adaptive Residual Minimization

Let $\vec{\lambda}_i$ be a characteristic speed evaluated at vertex i. Then, by setting $\vec{v}_i = \vec{\lambda}_i$, we obtain

$$\frac{dS_T}{dt} = \frac{1}{2}\sum_i \vec{\lambda}_i \cdot \vec{n}_i. \tag{6}$$

[*] This can be derived directly from the integration of the divergence of $\vec{v}$ assuming $\vec{v}$ varies linearly within the element.

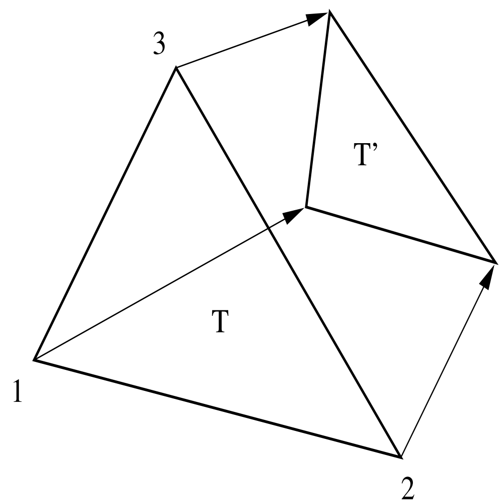

Figure 1: Element in compression

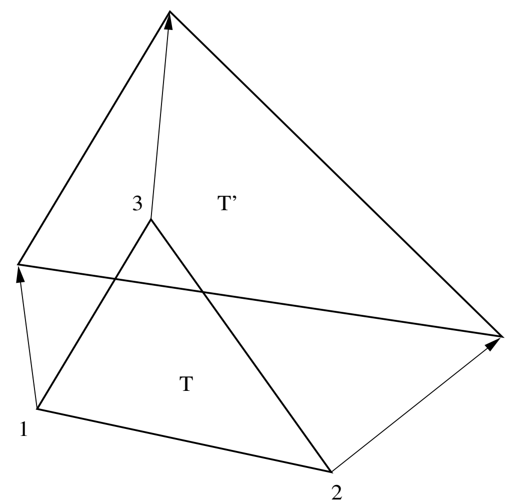

Figure 2: Element in expansion

This is the area change due to the convection of the element in the characteristic field. Then, this quantity tells us by its sign if the characteristics are diverging or converging, or equivalently expansion or compression. That is to say, the element is being compressed if

$$\frac{dS_T}{dt} < 0 \tag{7}$$

whereas being expanded if

$$\frac{dS_T}{dt} > 0. \tag{8}$$

Of course, it can vanish also, e.g. in locally linear fields.

In the case of negative area change, the element is in a compression wave. But there is a possibility that it might lie on a shock. We seek the conditions that identify such a special case. A main difference between a shock and compression would be that the element area becomes negative more quickly in the case of a shock element than a compression element. So, the time required for the element area to vanish, t', would be a useful measure. This can be estimated by setting $S_{T'} = 0$ in (4).

$$t' = -\frac{S_T}{\frac{1}{2}\sum_i \vec{\lambda}_i \cdot \vec{n}_i}. \tag{9}$$

Hence the element lies on a shock if this is small. But how small should it be? We don't know yet.

Consider the residual with the parameter α. We know that $\alpha = 1$ is suitable for shock capturing and characteristic recognition, and that $\alpha = 0$ is suitable for removing expansion shocks. It is therefore desirable to use different residuals for different elements depending on the nature of the flow solution. A strategy would be to compute an initial solution by some α (the same everywhere), compute the area change (6) for each element based on the initial solution, choosing suitable α (1 if negative; 0 if positive), and then minimize the new residuals to get the final solution. This indeed works, especially effective for automatically remove expansion shocks. And also, this way we can capture a grid aligned shock perfectly. In other situations, we can capture a shock perfectly by allowing the grid to move.

3 Time Step Restriction for Grid Movement

We use Gauss-Seidel iteration to minimize the residuals, and so consider all the contributions from the triangles that share a vertex. We write the update of the coordinate for vertex j as

$$\delta\vec{x}_j = -\Delta t_j\, \mathcal{G}_j \tag{10}$$

where Δt_j is a nodal time step and $\mathcal{G}_j$ is the gradient given by

$$\mathcal{G}_j = \sum_{T \in \{T_j\}} \left(\frac{\partial F_T}{\partial \vec{x}_j} \right)^t \tag{11}$$

Then, substitute this change into (4) to get

$$S_{T'} - S_T = -\frac{\Delta t_j}{2} \left(\mathcal{G}_j \cdot \vec{n}_T \right) \tag{12}$$

for $T \in \{T_j\}$ where $\vec{n}_T$ is the scaled inward normal of the edge opposite to the node j. Note that we have assumed that only the vertex j will move. We then require all the triangle area to remain positive.

$$S_{T'} = S_T - \frac{\Delta t_j}{2} \left(\mathcal{G}_j \cdot \vec{n}_T \right) > 0 \qquad \forall \; T \in \{T_j\} \tag{13}$$

or written in terms of the relative change

$$\frac{S_{T'} - S_T}{S_T} = -\Delta t_j \left[\frac{\mathcal{G}_j \cdot \vec{n}_T}{2 \, S_T} \right] > 0 \qquad \forall \; T \in \{T_j\}. \tag{14}$$

This leads to the following condition.

$$\nu_j \equiv \Delta t_j \, \max_{\{T_j\}} \left[\frac{\mathcal{G}_j \cdot \vec{n}_T}{2 \, S_T} \right] < 1 \tag{15}$$

where ν_j represents the largest relative change (reduction) in the area among the triangles that share the vertex j. Therefore, the time step can be chosen by

$$\Delta t_j = \frac{\nu_j}{\max_{\{T_j\}} \left[\frac{\mathcal{G}_j \cdot \vec{n}_T}{2 \, S_T} \right]} \qquad (0 \leq \nu_j < 1). \tag{16}$$

Hence, for the choice $\nu_j = 0.1$ for example, the largest reduction in the area is always 10%. But a more practical formula would be

$$\Delta t_j = \begin{cases} \dfrac{\nu_j}{\max_{\{T_j\}} \left[\frac{\mathcal{G}_j \cdot \vec{n}_T}{2 \, S_T} \right]} & \max_{\{T_j\}} \left[\frac{\mathcal{G}_j \cdot \vec{n}_T}{2 \, S_T} \right] > \nu_j \\ 1 & \max_{\{T_j\}} \left[\frac{\mathcal{G}_j \cdot \vec{n}_T}{2 \, S_T} \right] \leq \nu_j \end{cases} \tag{17}$$

i.e. if the largest change is smaller than the upper limit specified by ν_j, we do not magnify it to ν_j and let it change by that small amount. With the time step determined this way, negative volume is never created, or equivalently mesh tangling never occur.

