Averaging is Dissipation

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It is shown that cell-averaging introduces dissipation.

1 Basic Advection Schemes

We begin with a list of basic schemes for the advection equation:

$$\partial_t u + a \partial_x u = 0, \tag{1}$$

where a is a positive constant.

1.1 Upwind

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{a\Delta t}{h} \left(u_j - u_{j-1} \right) \\ &= u_j^n - \sigma \left(u_j - u_{j-1} \right). \end{aligned} \tag{2}$$

1.2 Lax-Wendroff

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{a\Delta t}{2h} \left(u_{j+1} - u_{j-1} \right) + \frac{a^2 \Delta t^2}{2h^2} \left(u_{j+1} - 2u_j + u_{j-1} \right) \\ &= u_j^n - \frac{\sigma}{2} \left(u_{j+1} - u_{j-1} \right) + \frac{\sigma^2}{2} \left(u_{j+1} - 2u_j + u_{j-1} \right). \end{aligned} \tag{3}$$

1.3 Beam-Warming

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{a\Delta t}{2h} \left(3u_j - 4u_{j-1} + u_{j-2} \right) + \frac{a^2 \Delta t^2}{2h^2} \left(u_j - 2u_{j-1} + u_{j-2} \right) \\ &= u_j^n - \frac{\sigma}{2} \left(3u_j - 4u_{j-1} + u_{j-2} \right) + \frac{\sigma^2}{2} \left(u_j - 2u_{j-1} + u_{j-2} \right). \end{aligned} \tag{4}$$

1.4 Fromm = Average of Lax-Wendroff and Beam-Warming

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{a\Delta t}{4h} \left(u_{j+1} + 3u_j - 5u_{j-1} + u_{j-2} \right) + \frac{a^2 \Delta t^2}{2h^2} \left(u_{j+1} - u_j - u_{j-1} + u_{j-2} \right) \\ &= u_j^n - \frac{\sigma}{2} \left(u_{j+1} + 3u_j - 5u_{j-1} + u_{j-2} \right) + \frac{\sigma^2}{2} \left(u_{j+1} - u_j - u_{j-1} + u_{j-2} \right). \end{aligned} \tag{5}$$

2 Lagrangian Schemes: Exact Convection and Cell Averaging

The basic schemes can be derived from reconstructed polynomials by following the original idea of Van Leer. Consider piecewise linear polynomials in the cells $j-1$ and j, respectively:

$$P_{j-1}(x) = u_{j-1} + p_{j-1}(x - x_{j-1}), \quad x \in [-3h/2, -h/2], \tag{6}$$

$$P_j(x) = u_j + p_j(x - x_j), \quad x \in [-h/2, h/2]. \tag{7}$$

The cell-averaged solution in the cell j at the new time level $t = t + \Delta t$ is found by exactly convecting the polynomials at the speed a for the time Δt, and computing the cell-average. The solution profile in the cell j after the exact convection corresponds to P_{j-1} over $x \in [-h/2 - a\Delta t, -h/2]$ and P_j over $x \in [-h/2, h/2 - a\Delta t]$. Then, the new cell-averaged solution is given by

$$u_j^{n+1} = \frac{1}{h}\left[\int_{-h/2-a\Delta t}^{-h/2} P_{j-1}(x)\,dx + \int_{-h/2}^{h/2-a\Delta t} P_j(x)\,dx\right] \tag{8}$$

$$= u_j^n - \sigma\left[(u_j - u_{j-1}) + \frac{h}{2}\{p_j - p_{j-1}\}\right] + \frac{\sigma^2 h}{2}\{p_j - p_{j-1}\}. \tag{9}$$

Note that dissipation has just been created by the averaging (projection). To see this, write the scheme (9) as

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}\left[f_{j+1/2} + f_{j-1/2}\right] + \frac{\sigma^2 h}{2}\{p_j - p_{j-1}\}. \tag{10}$$

where $f_{j+1/2}$ is the upwind flux evaluated by the linearly reconstructed solutions:

$$f_{j+1/2} = \frac{au_L + au_R}{2} - \frac{a}{2}(u_R - u_L), \tag{11}$$

$$u_L = u_j + \frac{h}{2}p_j, \quad u_R = u_{j+1} - \frac{h}{2}p_{j+1}, \tag{12}$$

and similarly for $f_{j-1/2}$. Note that the last term proportional to $(p_j - p_{j-1})$ is also a dissipation term, which can be easily seen by the fact that the basic schemes are reproduced by the following choices of the gradients:

$$p_j^{UW} = 0, \quad p_{j-1}^{UW} = 0, \tag{13}$$

$$p_j^{LW} = \frac{u_{j+1} - u_j}{h}, \quad p_{j-1}^{LW} = \frac{u_j - u_{j-1}}{h}, \tag{14}$$

$$p_j^{BW} = \frac{u_j - u_{j-1}}{h}, \quad p_{j-1}^{BW} = \frac{u_{j-1} - u_{j-2}}{h}, \tag{15}$$

$$p_j^{Fromm} = \frac{u_{j+1} - u_{j-1}}{2h}, \quad p_{j-1}^{Fromm} = \frac{u_j - u_{j-2}}{2h}. \tag{16}$$

Therefore, although the scheme is constructed based on the exact convection (no Riemann solvers such as the upwind flux), the averaging process introduces dissipation. And it is actually equivalent to the use of upwind flux (a Riemann solver).