# On Dissipation in CESE Method

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## 1 CESE Method for Linear Advection

Consider the space-time control-volumes, $CE_-$ and $CE_+$, shown in Figure 1. Given solution values and derivatives at $(x_{j-1}, t_n)$ and $(x_{j+1}, t_n)$, we consider computing the solution and the derivative at $(x_j, t_{n+1})$ by solving the linear advection equation:

$$u_t + f_x = 0, \tag{1}$$

where $f = au$ and $a$ is a constant advection speed. The solution and the derivative at $(x_j, t_{n+1})$ are determined such that the integral form of the advection equation is satisfied over the control volumes. Integrating the advection equation over each control volume by the midpoint rule, we obtain

$$F_- \equiv \int_{CE_-} (u_t + f_x) = \Delta t \left[ f_j^{n+1/2} - f_{j-1}^{n+1/2} \right] + \Delta x \left[ u_j^{n+1} - \frac{(u_x)_j^{n+1} \Delta x}{2} - u_{j-1/2}^n \right], \tag{2}$$

$$F_+ \equiv \int_{CE_+} (u_t + f_x) = \Delta t \left[ f_{j+1}^{n+1/2} - f_j^{n+1/2} \right] + \Delta x \left[ u_j^{n+1} + \frac{(u_x)_j^{n+1} \Delta x}{2} - u_{j+1/2}^n \right], \tag{3}$$

where

$$u_{j\pm 1/2}^n = u_{j\pm 1}^n \mp (u_x)_{j\pm 1}^n \frac{\Delta x}{2} \tag{4}$$

$$f_{j\pm 1}^{n+1/2} = a u_{j\pm 1}^{n+1/2} = a \left( u_{j\pm 1}^n + (u_t)_{j\pm 1}^n \frac{\Delta t}{2} \right) \tag{5}$$

$$= a \left( u_{j\pm 1}^n - (f_x)_{j\pm 1}^n \frac{\Delta t}{2} \right), \tag{6}$$

$$= a u_{j\pm 1}^n - a^2 (u_x)_{j\pm 1}^n \frac{\Delta t}{2}. \tag{7}$$

The evaluation of $f_j^{n+1/2}$ is left open at the moment. The solution and the derivative at the new time level are determined by solving the equations:

$$F_- = 0, \quad F_+ = 0. \tag{8}$$

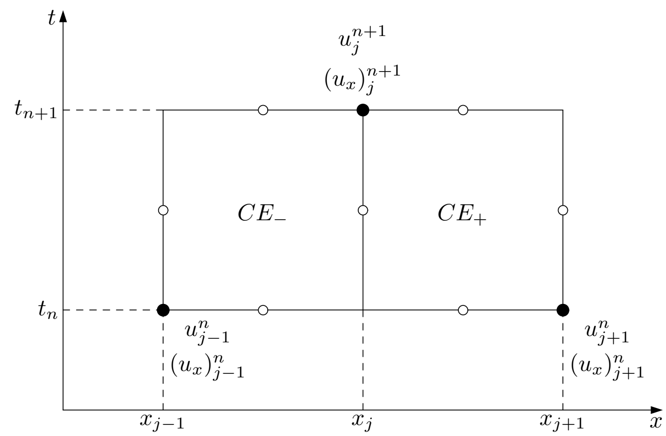


Figure 1: Space-time control volumes used to compute the solution and the derivative at $(x_j, t_{n+1})$.

The solution value can be determined without $f_j^{n+1/2}$ by solving the equation, $F_- + F_+ = 0$,

$$F_- + F_+ = \Delta t \left[ f_{j+1}^{n+1/2} - f_{j-1}^{n+1/2} \right] + \Delta x \left[ 2u_j^{n+1} - (u_{j-1/2}^n + u_{j+1/2}^n) \right] = 0, \tag{9}$$

thus giving

$$u_j^{n+1} = \frac{1}{2} \left[ u_{j-1/2}^n + u_{j+1/2}^n \right] - \frac{\Delta t}{2\Delta x} \left[ f_{j+1}^{n+1/2} - f_{j-1}^{n+1/2} \right]. \tag{10}$$

The derivative, $(u_x)_j^{n+1}$, is then determined uniquely by either $F_- = 0$ or $F_+ = 0$. We obtain from $F_- = 0$ and Equation (10)

$$F_- = f_j^{n+1/2} \Delta t + \frac{\Delta x}{2} \left[ u_{j+1/2}^n - u_{j-1/2}^n \right] - \frac{\Delta t}{2} \left[ f_{j+1}^{n+1/2} + f_{j-1}^{n+1/2} \right] - \frac{(u_x)_j^{n+1} \Delta x^2}{2} = 0, \tag{11}$$

and thus

$$(u_x)_j^{n+1} = \frac{u_{j+1/2}^n - u_{j-1/2}^n}{\Delta x} - \Delta t \, \frac{f_{j+1}^{n+1/2} - 2f_j^{n+1/2} + f_{j-1}^{n+1/2}}{\Delta x^2}, \tag{12}$$

which is a consistent approximation of the derivative of the advection equation, i.e., a governing equation for $u_x$:

$$(u_x)_t + (f_x)_x = 0. \tag{13}$$

In order to compute the derivative, we need to define the interface flux, $f_j^{n+1/2}$. Note that the integral conservation equations, $F_- = 0$ and $F_+ = 0$, are satisfied independently of the definition of $f_j^{n+1/2}$.

# 2 Evaluation of Interface Flux: $f_j^{n+1/2}$

## 2.1 Non-Dissipative Scheme

The interface flux can be evaluated by using the solution at $t = t_{n+1}$:

$$\begin{aligned} f_j^{n+1/2} &= a u_j^{n+1/2} && (14) \\ &= a\left(u_j^{n+1} - (u_t)_j^{n+1}\frac{\Delta t}{2}\right) && (15) \\ &= a\left(u_j^{n+1} + (f_x)_j^{n+1}\frac{\Delta t}{2}\right) && (16) \\ &= a u_j^{n+1} + a^2 (u_x)_j^{n+1}\frac{\Delta t}{2}. && (17) \end{aligned}$$

This choice corresponds to the $a$-scheme [**?**].

## 2.2 Dissipative Scheme via Riemann Solver (Upwind Flux)

At time level $n$, we have a piecewise linear representation of the solution independently at $x_{j-1}$ and $x_{j+1}$. Hence, there exists a gap at $(x_j, t_n)$. Denote the left and the right values at the gap by $u_j^-$ and $u_j^+$. The interface flux may be determined by solving the generalized Riemann problem between these values. For example, the upwind flux is given by

$$f_j^{n+1/2} = \frac{1}{2}\left[f_j^+ + f_j^-\right] - \frac{1}{2}|a|(u_j^+ - u_j^-), \tag{18}$$

where $f_j^- = a u_j^-$ and $f_j^+ = a u_j^+$. Note that the first term approximates the physical flux consistently while the second term is the dissipation term which is a quantity of $O(\Delta x^3)$.

## 2.3 Dissipative Scheme via q-Flux

In fact, any advective flux (not necessarily upwind) can be employed to determine the interface flux. Consider the flux of the form:

$$f_j^{n+1/2} = \frac{1}{2}\left[f_j^+ + f_j^-\right] - \frac{q}{2}(u_j^+ - u_j^-), \tag{19}$$

where $q$ is a parameter. This includes the following schemes:

$$\begin{aligned} \text{Upwind} &: \quad q = |a|, && (20) \\ \text{Lax-Friedrichs} &: \quad q = \frac{\Delta x}{\Delta t} = \frac{a}{\nu}, && (21) \\ \text{Lax-Wendroff} &: \quad q = a^2\frac{\Delta t}{\Delta x} = a\nu, && (22) \\ \text{FCT(LF-LW)} &: \quad q = \frac{a}{\nu}\left(1-\nu^2\right). && (23) \end{aligned}$$

 

The last choice is obtained by subtracting the dissipation term of the Lax-Wendroff flux from the Lax-Friedrichs flux. It is an FCT-type scheme: an anti-diffusive term added to the Lax-Friedrichs flux. It is still stable for $\nu \leq 1$.

### 2.4 Dissipative $a$-Scheme

Replacing the consistent term in the $q$-flux by the $a$-scheme, we obtain a dissipative $a$-scheme:

$$f_j^{n+1/2} = \left[ a u_j^{n+1} + a^2 (u_x)_j^{n+1} \frac{\Delta t}{2} \right] - \frac{q}{2}(u_j^+ - u_j^-). \tag{24}$$

For example, by using the dissipation of the FCT(LF-LW) flux, i.e., $q = \epsilon \frac{a}{\nu}\left(1 - \nu^2\right)$, where $\epsilon$ is a parameter, we obtain

$$f_j^{n+1/2} = \left[ a u_j^{n+1} + a^2 (u_x)_j^{n+1} \frac{\Delta t}{2} \right] - \frac{\epsilon a}{2\nu}\left(1 - \nu^2\right)(u_j^+ - u_j^-), \tag{25}$$

which leads to the $a$-$\epsilon$ scheme [**?**]. Of course, we may choose the dissipation term of any other advective flux.

This way of adding a dissipation to the $a$-scheme can be readily extended to the Euler equations. For example, we may add the dissipation term of the Roe flux or the Rusanov flux to the $a$-scheme for the Euler equations.

## 3 Remarks

1. Dissipation can be introduced through the interface flux, $f_j^{n+1/2}$. Adding a dissipation this way does not affect the conservation over the control volumes, $CE_-$ and $CE_-$, not to mention the union of them.

2. The $a$-$\epsilon$ scheme [**?**] can be considered as adding the FCT-type dissipation to the $a$-scheme. Other dissipative scheme may be generated by using other dissipation term instead of the FCT-type dissipation.

3. A finite-volume person would consider the solution at time level $n$ as a piecewise linear solution with gaps at interfaces, and then go immediately for a Riemann solver or the $q$-flux without any doubt in evaluating the interface flux. The key feature of CESE method, I think, is the use of the solution at time level $n+1$ to evaluate the interface flux: it seems the most natural procedure in the CESE framework.

4. Analysis needs to be performed (or may have already been done) to compare properties of the resulting scheme for each dissipation term employed.

 