Derivation of Optimal Length Scale for Hyperbolic Diffusion Scheme

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October 20, 2016

Abstract

This note gives a detailed derivation of the optimal length scale for the hyperbolic diffusion scheme in Ref.[1]

1 Problem Statement

In Ref.[1], the following Fourier transformed first-order hyperbolic diffusion scheme was obtained on a Cartesian grid:

$$\frac{d\mathbf{U}_0}{dt} = \mathbf{M}\mathbf{U}_0, \tag{1}$$

where $\mathbf{M}$ is given for smooth components as

$$\mathbf{M} = \begin{bmatrix} -\dfrac{\nu\beta^2}{2hL_r} & \dfrac{i\nu\beta_x}{h} & \dfrac{i\nu\beta_y}{h} \\ \dfrac{i\nu\beta_x}{hL_r^2} & -\dfrac{\nu}{L_r^2} & 0 \\ \dfrac{i\nu\beta_y}{hL_r^2} & 0 & -\dfrac{\nu}{L_r^2} \end{bmatrix}, \tag{2}$$

where $\beta^2 = \beta_x^2 + \beta_y^2$, $h = L/N$, L is a reference length scale (e.g., the domain size) that is resolved by N cells, and N is the number of grid spacings (i.e., the grid has $N+1$ nodes). The eigenvalues are given by

$$-\frac{\nu}{L_r^2}, \quad -\frac{\nu}{2L_r^2}\left(1 + \frac{\beta^2}{2h} \pm \sqrt{1 - \frac{L_r\beta^2}{h} + \frac{L_r^2\beta^2}{4h^2}(\beta^2 - 16)}\right). \tag{3}$$

An optimal length scale was derived in Ref.[1] by requiring the expression inside the square root to be nonpositive, so that the Fourier mode propagates rather than is purely damped. The derivation is presented in the next section.

2 Derivation

We seek L_r that satisfies

$$1 - \frac{L_r\beta^2}{h} + \frac{L_r^2\beta^2}{4h^2}(\beta^2 - 16) \leq 0, \tag{4}$$

which we write

$$f(L_r) \leq 0, \quad f(L_r) = 1 - \frac{L_r\beta^2}{h} + \frac{L_r^2\beta^2}{4h^2}(\beta^2 - 16). \tag{5}$$

The function $f(L_r)$ can be factored as

$$f(L_r) = \frac{1}{4h^2}\left[(\beta - 4)\beta L_r - 2h\right]\left[(\beta + 4)\beta L_r - 2h\right]. \tag{6}$$

Solving $f(L_r) = 0$ for L_r, we obtain

$$L_r = \frac{2h}{(\beta - 4)\beta}, \quad \frac{2h}{(\beta + 4)\beta}. \tag{7}$$

For smooth components, e.g., $(\beta_x, \beta_y) \in (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$, we have

$$\beta < 4, \tag{8}$$

and therefore,

$$\frac{2h}{(\beta - 4)\beta} < 0 < \frac{2h}{(\beta + 4)\beta}. \tag{9}$$

Hence, we have $f(L_r) \leq 0$ for

$$L_r \leq \frac{2h}{(\beta - 4)\beta} \leq 0, \quad L_r \geq \frac{2h}{(\beta + 4)\beta} \geq 0, \tag{10}$$

but since $L_r > 0$, we are left with

$$L_r \geq \frac{2h}{(\beta + 4)\beta}, \tag{11}$$

(which indicates that the inequality sign in Ref.[1] is actually wrong). Since β varies from $\beta_{min} = \pi h/L = \pi/N$ to $\beta_{max} = \pi$, the condition is satisfied for all values of β if we set

$$L_r = \frac{2h}{(\pi h/L + 4)\pi h/L} = \frac{2}{(\pi/N + 4)\pi} L \quad \geq \frac{2h}{(\beta + 4)\beta}, \quad \text{for} \quad \beta \geq \pi/N. \tag{12}$$

For small h, or equivalently for large N, we can set

$$L_r = \frac{L}{2\pi} \quad \geq \frac{2}{(\pi/N + 4)\pi} L \quad \geq \frac{2h}{(\beta + 4)\beta}, \quad \text{for} \quad \beta \geq \pi/N. \tag{13}$$

Note: By assumption, the length L is taken to be a reference length resolved by a sufficient number of cells. If the target equation has been nondimensionalized by a reference length scale L, the length scale L_r is set as $L_r/L = 1/(2\pi)$.

References

[1] H. Nishikawa. First-, second-, and third-order finite-volume schemes for diffusion. *J. Comput. Phys.*, 256:791–805, 2014.