# On Entropy Generation and Dissipation

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September 2005

## 1 Entropy Generation

Consider the scalar conservation law with the diffusion term,

$$u_t + f_x = \epsilon u_{xx} \tag{1}$$

Define $v = \partial U / \partial u$ where $U$ is a non-increasing entropy function. Then, multiplying (1) by $v$, we obtain the equation for $U$.

$$vu_t + vf_x = \epsilon v u_{xx} \tag{2}$$

$$\longrightarrow \quad U_t + F_x = \epsilon(U_{xx} - v_x u_x) \tag{3}$$

We integrate this over the interval $x = [x_L, x_R]$ to obtain

$$\bar{U}_t + [F] = \epsilon[U_x] - \epsilon \int_{x_L}^{x_R} v_x u_x \, dx \tag{4}$$

where $\bar{U} = \int_{x_L}^{x_R} U \, dx$ and $[\;] = ()_R - ()_L$. Observe that as $\epsilon \to 0$, $\epsilon[U_x] \to 0$, but the second term does not vanish if there is a discontinuity inside the interval. Note in particular that

$$-\epsilon \int_{x_L}^{x_R} v_x u_x \, dx = -\epsilon \int_{x_L}^{x_R} v_x \frac{\partial u}{\partial v} v_x \, dx \leq 0 \tag{5}$$

provided $\frac{\partial u}{\partial v} \geq 0$. Hence, this is the term that generates entropy from inside a discontinuity. For small $\epsilon$, therefore, we may write

$$\bar{U}_t + [F] = -\epsilon \int_{x_L}^{x_R} v_x \frac{\partial u}{\partial v} v_x \, dx \tag{6}$$

This is usually written as an inequality,

$$\bar{U}_t + [F] \leq 0 \tag{7}$$

## 2 Dissipation

Now consider a numerical flux functin $f^*$ defined by

$$f^* = f_c - \frac{1}{2}Q[v] \tag{8}$$

Then, the rate of change of the entropy by this flux is

$$\frac{\partial \hat{U}}{\partial t} = [v]f^* - [vf] \tag{9}$$

where $\hat{U}$ is an integral value of the numerical solution over the cell $[x_L, x_R]$. Then, inserting $f^*$ into this, we obtain

$$\frac{\partial \hat{U}}{\partial t} = [v]f_c - [vf] - \frac{1}{2}[v]Q[v] \tag{10}$$

and if we define $f_c$ such that $[v]f_c - [vf] = -[F]$ (entropy conserving flux),

$$\frac{\partial \hat{U}}{\partial t} = -[F] - \frac{1}{2}[v]Q[v] \tag{11}$$

Now, requiring that this matches (6), we obtain

$$\frac{1}{2}[v]Q[v] = \epsilon \int_{x_L}^{x_R} v_x \frac{\partial u}{\partial v} v_x \, dx. \tag{12}$$

A crude approximation to this would be

$$\frac{1}{2}[v]Q[v] = \frac{\epsilon}{\Delta x}[v]\frac{\partial u}{\partial v}[v] \tag{13}$$

so that

$$Q = \frac{2\epsilon}{\Delta x}\frac{\partial u}{\partial v} \tag{14}$$

The flux function is therefore

$$f^* = f_c - \frac{\epsilon}{\Delta x}\frac{\partial u}{\partial v}[v] \tag{15}$$

## 3 Burgers' Equation

In the case of Burgers' equation, we have $v = 2u$ and $\frac{\partial u}{\partial v} = \frac{1}{2}$. So,

$$f^* = f_c - \frac{\epsilon}{\Delta x}[u] \tag{16}$$

To determine $\epsilon$, consider the exact solution to the viscous Burgers' equation,

$$u = u_R - \frac{1}{2}[u]\left\{1 - \tanh\left(-\frac{[u](x - st)}{4\epsilon}\right)\right\} \tag{17}$$

where $s = \frac{u_R + u_L}{2}$, which describes a viscous shock layer traveling at the speed $s$. By using this, we can estimate the width of the shock layer, $\ell$. We fit a linear function through the center of the shock layer, using the exact slope there, and then compute the distance where the solution becomes $u_R$ or $u_L$. The result is

$$\ell = -\frac{8\epsilon}{[u]} \tag{18}$$

Note that across a shock $[u] < 0$, so that $\ell > 0$. Now, let $\ell = k\Delta x$ where $k$ is a constant, then solve for $\epsilon$ to get

$$\epsilon = -\frac{[u]k\Delta x}{8} \tag{19}$$

So, we have

$$f^* = f_c + k\frac{[u]^2}{8} \tag{20}$$

This dissipation term may be used alone or as an addition to the existing dissipation to provide more dissipation in case of shocks. So it should be switched off when there is no shock. For example,

$$f^* = f_c + k\frac{\min(0, [u])}{8}[u] \tag{21}$$

will do it.

# 4 Extension to Systems

Everythig is carried over to the system case. For system of equations, we have

$$\mathbf{f}^* = \mathbf{f}_c - \frac{\epsilon}{\Delta x}\frac{\partial \mathbf{u}}{\partial \mathbf{v}}[\mathbf{v}] \tag{22}$$

which gives the entropy change in the form,

$$\frac{\partial \hat{U}}{\partial t} = -[F] - \frac{\epsilon}{\Delta x}[\mathbf{v}]^T\frac{\partial \mathbf{u}}{\partial \mathbf{v}}[\mathbf{v}] \tag{23}$$

So, the entropy is generated correctly if the matrix $\frac{\partial \mathbf{u}}{\partial \mathbf{v}}$ is positive definite. In the case of the Euler equations, with a proper scaling, we have the indentity,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{v}} = \mathbf{R}\mathbf{R}^T \tag{24}$$

and therefore it provides a correct entropy generation. Using this identity, we can rewrite the flux function as

$$\mathbf{f}^* = \mathbf{f}_c - \frac{1}{2}\mathbf{R}\lambda\mathbf{R}^T[\mathbf{v}] \tag{25}$$

where

$$\lambda = \frac{2\epsilon}{\Delta x} \tag{26}$$

which is in the form of a one-wave approximation of a more general flux function

$$\mathbf{f}^* = \mathbf{f}_c - \frac{1}{2}\mathbf{R}\,|\Lambda|\,\mathbf{R}^T[\mathbf{v}] \tag{27}$$

where $|\Lambda|$ is a diagonal matrix. This is a pleasing result. This makes it very simple to use the dissipation term in (25) as an addition to the existing dissipation because it can be done in the form of modified wave speeds, i.e.

$$\mathbf{f}^* = \mathbf{f}_c - \frac{1}{2}\mathbf{R}\,|\Lambda|^*\,\mathbf{R}^T[\mathbf{v}] \tag{28}$$

where

$$|\Lambda|^* = |\Lambda| + \lambda\mathbf{I} = |\Lambda| + \frac{2\epsilon}{\Delta x}\mathbf{I} \tag{29}$$

$\epsilon$ must be determined in some way. Also, the extra dissipation should be terned off away from shocks. These are not as simple as in the Burgers case.