Forms of the Euler Equations

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1 Conservative Variables to Primitive Variables

We begin with the Euler equations in the conservative form.

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial x} = 0 \tag{1}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{bmatrix} \tag{2}$$

where ρ is the density, u and v are the velocity components in the x and y direction, respectively, and p is the static pressure. The specific energy and enthalpy are given by

$$E = \frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2) \tag{3}$$

$$H = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2). \tag{4}$$

The vector of primitive variables is given by

$$\mathbf{W} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix} \tag{5}$$

which is linked with the consistent and conservative variables through the transformations

$$\partial \mathbf{U} = \mathbf{T}^{-1} \partial \mathbf{W} \tag{6}$$

where

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{1}{2} q^2 & \rho\, u & \rho\, v & \frac{1}{\gamma - 1} \end{bmatrix}. \tag{7}$$

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Its inverse is given by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{\gamma-1}{2}q^2 & u(\gamma-1) & v(\gamma-1) & \gamma-1 \end{bmatrix}. \tag{8}$$

Now, we can transform the conservation form into the primitive variable form by multiplying (1) by ${\mathbf{T_{uw}}}^{-1}$ from the left.

$$\mathbf{T}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{T}\left(\frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial x}\right) = 0 \tag{9}$$

$$\mathbf{T}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{T}\left(\mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial x}\right) = 0 \tag{10}$$

$$\mathbf{T}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{T}\left(\mathbf{A}\mathbf{T}^{-1}\mathbf{T}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\mathbf{T}^{-1}\mathbf{T}\frac{\partial \mathbf{U}}{\partial x}\right) = 0 \tag{11}$$

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\frac{\partial \mathbf{W}}{\partial x} + \mathbf{T}\mathbf{B}\mathbf{T}^{-1}\frac{\partial \mathbf{W}}{\partial x} = 0 \tag{12}$$

where $\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$, $\mathbf{B} = \frac{\partial \mathbf{G}}{\partial \mathbf{U}}$, and we thus find

$$\mathbf{A_w} \equiv \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{bmatrix} \tag{13}$$

$$\mathbf{B_w} \equiv \mathbf{T}\mathbf{B}\mathbf{T}^{-1} = \begin{bmatrix} v & 0 & p & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} \\ 0 & 0 & \gamma p & v \end{bmatrix} \tag{14}$$

where $M = q/a$ is the Mach number. For simplicity, we write the Euler equations in terms of the natural coordinates: the streamline and its normal.

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A_{ws}}\frac{\partial \mathbf{W}}{\partial s} + \mathbf{B_{wn}}\frac{\partial \mathbf{W}}{\partial n} = 0 \tag{15}$$

where

$$\mathbf{A_{ws}} = \mathbf{A_w}\cos\theta + \mathbf{B_w}\sin\theta = \begin{bmatrix} q & \frac{\rho u}{q} & \frac{\rho v}{q} & 0 \\ 0 & q & 0 & \frac{u}{\rho q} \\ 0 & 0 & q & \frac{v}{\rho q} \\ 0 & \frac{\gamma u p}{q} & \frac{\gamma v p}{q} & q \end{bmatrix} \tag{16}$$

$$\mathbf{B_{wn}} = \mathbf{B_w}\cos\theta - \mathbf{A_w}\sin\theta = \begin{bmatrix} 0 & -\frac{\rho v}{q} & \frac{\rho u}{q} & 0 \\ 0 & 0 & 0 & -\frac{v}{\rho q} \\ 0 & 0 & 0 & \frac{u}{\rho q} \\ 0 & -\frac{\gamma v p}{q} & \frac{\gamma u p}{q} & 0 \end{bmatrix}. \tag{17}$$

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2 Symmetrizing Variables

The vector of the symmetrizing variables is

$$\partial \mathbf{U_m} = \begin{bmatrix} \frac{\partial p}{\rho a} \\ \partial q \\ q\partial\theta \\ \partial s \end{bmatrix} \tag{18}$$

which is linked with the primitive variables through the transformations

$$\partial \mathbf{U_m} = \mathbf{T_m} \partial \mathbf{W} \tag{19}$$

where

$$\mathbf{T_m} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\rho a} \\ 0 & \frac{u}{q} & \frac{v}{q} & 0 \\ 0 & -\frac{v}{q} & \frac{u}{q} & 0 \\ -a^2 & 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

Its inverse is

$$\mathbf{T_m}^{-1} = \begin{bmatrix} \frac{\rho}{a} & 0 & 0 & -\frac{1}{a^2} \\ 0 & \frac{u}{q} & -\frac{v}{q} & 0 \\ 0 & \frac{v}{q} & \frac{u}{q} & 0 \\ \rho a & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

Now, we can transform the conservation form into the primitive variable form by multiplying (1) by $\mathbf{T_m}$ from the left.

$$\mathbf{T_m}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T_m}\left(\mathbf{A_{ws}}\frac{\partial \mathbf{W}}{\partial s} + \mathbf{B_{wn}}\frac{\partial \mathbf{W}}{\partial n}\right) = 0 \tag{22}$$

$$\mathbf{T_m}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T_m}\left(\mathbf{A_{ws}}\mathbf{T_m}^{-1}\mathbf{T_m}\frac{\partial \mathbf{W}}{\partial s} + \mathbf{B_{wn}}\mathbf{T_m}^{-1}\mathbf{T_m}\frac{\partial \mathbf{W}}{\partial n}\right) = 0 \tag{23}$$

$$\frac{\partial \mathbf{U_m}}{\partial t} + \mathbf{T_m}\mathbf{A_{ws}}\mathbf{T_m}^{-1}\frac{\partial \mathbf{U_m}}{\partial s} + \mathbf{T_m}\mathbf{B_{wn}}\mathbf{T_m}^{-1}\frac{\partial \mathbf{U_m}}{\partial n} = 0 \tag{24}$$

where

$$\mathbf{A_{ms}} = \mathbf{T_m}\mathbf{A_{ws}}\mathbf{T_m}^{-1} = \begin{bmatrix} q & a & 0 & 0 \\ a & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \tag{25}$$

$$\mathbf{B_{mn}} = \mathbf{T_m}\mathbf{B_{wn}}\mathbf{T_m}^{-1} = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

In Cartesian coordinates, we obtain

$$\mathbf{T_m}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T_m}\left(\mathbf{A_w}\frac{\partial \mathbf{W}}{\partial x} + \mathbf{B_w}\frac{\partial \mathbf{W}}{\partial y}\right) = 0 \tag{27}$$

$$\mathbf{T_m}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T_m}\left(\mathbf{A_w}\mathbf{T_m}^{-1}\mathbf{T_m}\frac{\partial \mathbf{W}}{\partial x} + \mathbf{B_w}\mathbf{T_m}^{-1}\mathbf{T_m}\frac{\partial \mathbf{W}}{\partial y}\right) = 0 \tag{28}$$

$$\frac{\partial \mathbf{U_m}}{\partial t} + \mathbf{T_m}\mathbf{A_w}\mathbf{T_m}^{-1}\frac{\partial \mathbf{U_m}}{\partial x} + \mathbf{T_m}\mathbf{B_w}\mathbf{T_m}^{-1}\frac{\partial \mathbf{U_m}}{\partial y} = 0 \tag{29}$$

where

$$
\begin{aligned}
\mathbf{A_m} &= \mathbf{T_m A_w T_m}^{-1} = \begin{bmatrix} u & u/M & -v/M & 0 \\ u/M & u & 0 & 0 \\ -v/M & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} && (30) \\
\mathbf{B_m} &= \mathbf{T_m B_w T_m}^{-1} = \begin{bmatrix} v & v/M & u/M & 0 \\ v/M & v & 0 & 0 \\ u/M & 0 & v & 0 \\ 0 & 0 & 0 & v \end{bmatrix}. && (31)
\end{aligned}
$$

3 Another Symmetrizing Variables

The state vector

$$
\partial \mathbf{U_c} = \begin{bmatrix} \frac{\partial p}{\rho a} \\ \partial u \\ \partial v \\ \partial s \end{bmatrix} \tag{32}
$$

also symmetrizes the Euler equations. The transformation matrix is given by

$$
\mathbf{T_c} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\rho a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a^2 & 0 & 0 & 1 \end{bmatrix}. \tag{33}
$$

where

$$
\partial \mathbf{U_c} = \mathbf{T_c} \partial \mathbf{W} \tag{34}
$$

Its inverse is

$$
\mathbf{T_c}^{-1} = \begin{bmatrix} \frac{\rho}{a} & 0 & 0 & -\frac{1}{a^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \rho a & 0 & 0 & 0 \end{bmatrix}. \tag{35}
$$

Now, we can transform the primitive form into the primitive variable form by multiplying (1) by $\mathbf{T_c}$ from the left.

$$
\begin{aligned}
\mathbf{T_c}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T_c}\left(\mathbf{A_{ws}}\frac{\partial \mathbf{W}}{\partial s} + \mathbf{B_{wn}}\frac{\partial \mathbf{W}}{\partial n}\right) &= 0 && (36) \\
\mathbf{T_c}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T_c}\left(\mathbf{A_{ws}}\mathbf{T_c}^{-1}\mathbf{T_c}\frac{\partial \mathbf{W}}{\partial s} + \mathbf{B_{wn}}\mathbf{T_c}^{-1}\mathbf{T_c}\frac{\partial \mathbf{W}}{\partial n}\right) &= 0 && (37) \\
\frac{\partial \mathbf{U_c}}{\partial t} + \mathbf{T_c A_{ws} T_c}^{-1}\frac{\partial \mathbf{U_c}}{\partial s} + \mathbf{T_c B_{wn} T_c}^{-1}\frac{\partial \mathbf{U_c}}{\partial n} &= 0 && (38)
\end{aligned}
$$

where

$$
\mathbf{A_{cs}} = \mathbf{T_c A_{ws} T_c}^{-1} = \begin{bmatrix} q & u/M & v/M & 0 \\ u/M & q & 0 & 0 \\ v/M & 0 & q & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \tag{39}
$$

$$\mathbf{B_{cn}} = \mathbf{T_c}\mathbf{B_{wn}}\mathbf{T_c}^{-1} = \begin{bmatrix} 0 & -v/M & u/M & 0 \\ -v/M & 0 & 0 & 0 \\ u/M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{40}$$

Similarly, in Cartesian coordinates, we obtain

$$\mathbf{T_c}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T_c}\left(\mathbf{A_w}\frac{\partial \mathbf{W}}{\partial x} + \mathbf{B_w}\frac{\partial \mathbf{W}}{\partial y}\right) = 0 \tag{41}$$

$$\mathbf{T_c}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{T_c}\left(\mathbf{A_w}\mathbf{T_c}^{-1}\mathbf{T_c}\frac{\partial \mathbf{W}}{\partial x} + \mathbf{B_w}\mathbf{T_c}^{-1}\mathbf{T_c}\frac{\partial \mathbf{W}}{\partial y}\right) = 0 \tag{42}$$

$$\frac{\partial \mathbf{U_c}}{\partial t} + \mathbf{T_c}\mathbf{A_w}\mathbf{T_c}^{-1}\frac{\partial \mathbf{U_c}}{\partial x} + \mathbf{T_c}\mathbf{B_w}\mathbf{T_c}^{-1}\frac{\partial \mathbf{U_c}}{\partial y} = 0 \tag{43}$$

where

$$\mathbf{A_c} = \mathbf{T_c}\mathbf{A_w}\mathbf{T_c}^{-1} = \begin{bmatrix} u & a & 0 & 0 \\ a & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \tag{44}$$

$$\mathbf{B_c} = \mathbf{T_c}\mathbf{B_w}\mathbf{T_c}^{-1} = \begin{bmatrix} v & 0 & a & 0 \\ 0 & v & 0 & 0 \\ a & 0 & v & 0 \\ 0 & 0 & 0 & v \end{bmatrix}. \tag{45}$$

4 Dimensionally Consistent Variables

The state vector

$$\partial \mathbf{U_p} = \begin{bmatrix} \partial p \\ \rho q \partial q \\ \rho q^2 \partial \theta \\ \partial s \end{bmatrix} \tag{46}$$

also symmetrizes the Euler equations. The transformation matrix is given by

$$\mathbf{T_p} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \rho u & \rho v & 0 \\ 0 & -\rho v & \rho u & 0 \\ -a^2 & 0 & 0 & 1 \end{bmatrix}. \tag{47}$$

where

$$\partial \mathbf{U_p} = \mathbf{T_p}\partial \mathbf{W} \tag{48}$$

Its inverse is

$$\mathbf{T_p}^{-1} = \begin{bmatrix} \frac{1}{a^2} & 0 & 0 & -\frac{\rho}{a^2} \\ 0 & \frac{u}{\rho q^2} & -\frac{v}{\rho q^2} & 0 \\ 0 & \frac{v}{\rho q^2} & \frac{u}{\rho q^2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{49}$$

Now, we can transform the primitive form into the primitive variable form by multiplying (1) by $\mathbf{T_c}$ from the left.

$$\frac{\partial \mathbf{U_p}}{\partial t} + \mathbf{T_p}\mathbf{A_{ws}}\mathbf{T_p}^{-1}\frac{\partial \mathbf{U_p}}{\partial s} + \mathbf{T_p}\mathbf{B_{wn}}\mathbf{T_p}^{-1}\frac{\partial \mathbf{U_p}}{\partial n} = 0 \tag{50}$$

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where

$$\mathbf{A_{ps}} = \mathbf{T_p A_{ws} T_p}^{-1} = \begin{bmatrix} q & a^2/q & 0 & 0 \\ q & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \tag{51}$$

$$\mathbf{B_{pn}} = \mathbf{T_p B_{wn} T_p}^{-1} = \begin{bmatrix} 0 & 0 & a^2/q & 0 \\ 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{52}$$

Similarly, in Cartesian coordinates, we obtain

$$\mathbf{A_p} = \mathbf{T_p A_w T_p}^{-1} = \begin{bmatrix} u & u/M^2 & -v/M^2 & 0 \\ u & u & 0 & 0 \\ -v & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix} \tag{53}$$

$$\mathbf{B_p} = \mathbf{T_p B_w T_p}^{-1} = \begin{bmatrix} v & v/M^2 & u/M^2 & 0 \\ v & v & 0 & 0 \\ u & 0 & v & 0 \\ 0 & 0 & 0 & v \end{bmatrix}. \tag{54}$$

References

[1] Roe, P. L., Fluctuation Splitting Schemes on Optimal Grids, AIAA Paper 97-2032, June 1997.

[2] Nishikawa, H., Rad, M., Roe, P. L., Grids and Solutions from Residual Minimization, ICCFD Proceedings, Kyoto. Springer-Verlag, 2000.

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