An extended version is available in
AIAA Paper 2010-5093 (download at http://www.hiroakinishikawa.com)
which includes the formulas for all types of elements: quads, triangles, tetrahedra, hexa, prism, and pyramid.

Finite-Volume Integration Forumulas

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1 Finite-Volume Integrations for a Single Element

Here we attempt to derive edge-based finite-volume formulas exact for linear functions for some typical elements used in the finite-volume method. We begin by integrating

$$\int_{dual} \operatorname{grad} u \; dV, \tag{1}$$

over a dual control volume around a node in a single element, exactly for a linear function, $u(x, y, z)$. We arrange the result into an edge-based finite-volume form, i.e., the sum over edges of the product of a directed area and a solution value at the midpoint of each edge, approximating the alternative form of (1):

$$\oint_{\partial(dual)} u \, \mathbf{n} dS, \tag{2}$$

where $\partial(dual)$, $\mathbf{n}$, and dS denote the boundary of the dual volume, the outward unit normal vector, and the infinitesimal measure of the boundary. The final result will be an edge-based finite-volume integral formula that is exact for a linear function, u.

1.1 Triangles

For a triangular element, we define the dual control volume for a node within the element by connecting the node, the midpoints, and the centroid of the element (see Figure 1). For such a dual volume, the volume of the dual, V_{dual}, is precisely $\frac{1}{3}$ of the volume of the element, V:

$$V_{dual} = \frac{1}{3} V. \tag{3}$$

Also, if u is a linear function, the gradient is a global constant and can be uniquely expressed in terms of the nodal values over the element:

$$\operatorname{grad} u = \frac{1}{2V} \sum_{i=1}^{3} u_i \, \mathbf{n}_i, \tag{4}$$

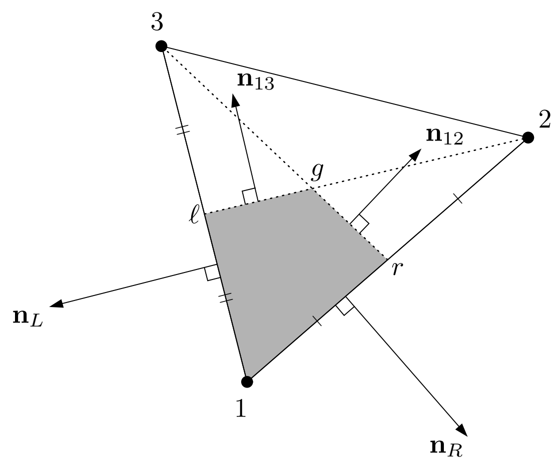

Figure 1: Dual control volume around 1 and scaled normals.

where u_i is the solution value at the node i and $\mathbf{n}_i$ is the scaled inward normal of the edge opposite to the node i (see Figure 2). Now we can evaluate the integral (1) as follows:

$$\int_{dual} \mathrm{grad}\, u\, dV = (\mathrm{grad}\, u) \times V_{dual}, \tag{5}$$

$$= (\mathrm{grad}\, u) \times \frac{1}{3} V, \tag{6}$$

$$= \frac{1}{6} \sum_{i=1}^{3} u_i\, \mathbf{n}_i, \tag{7}$$

$$= \frac{1}{6} \left(u_1\, \mathbf{n}_1 + u_2\, \mathbf{n}_2 + u_3\, \mathbf{n}_3 \right). \tag{8}$$

Substituting the following geometric identities,

$$\mathbf{n}_1 = 3(\mathbf{n}_{12} + \mathbf{n}_{13}) + 5(\mathbf{n}_L + \mathbf{n}_R), \tag{9}$$

$$\mathbf{n}_2 = 3\mathbf{n}_{12} + \mathbf{n}_L, \tag{10}$$

$$\mathbf{n}_3 = 3\mathbf{n}_{13} + \mathbf{n}_R, \tag{11}$$

we convert (8) into an edge-based finite-volume formula:

$$\int_{dual} \mathrm{grad}\, u\, dV = \frac{u_1 + u_3}{2} \mathbf{n}_{13} + \frac{u_1 + u_2}{2} \mathbf{n}_{12} + \left(\frac{5}{6} u_1 + \frac{1}{6} u_3 \right) \mathbf{n}_L + \left(\frac{5}{6} u_1 + \frac{1}{6} u_2 \right) \mathbf{n}_R. \tag{12}$$

This is the edge-based finite-volume integration formula that is exact for any linear function, u. It is nice that the boundary contributions over the edges 1ℓ and $1r$ are evaluated with the same set of weights, 5/6 and 1/6, without reference to the third node in each case. This allows us to evaluate the boundary integrals by working with boundary edges only (no need to access the element information). Also note that the average values in the first two terms can be replaced by any other interpolation formula (e.g.,

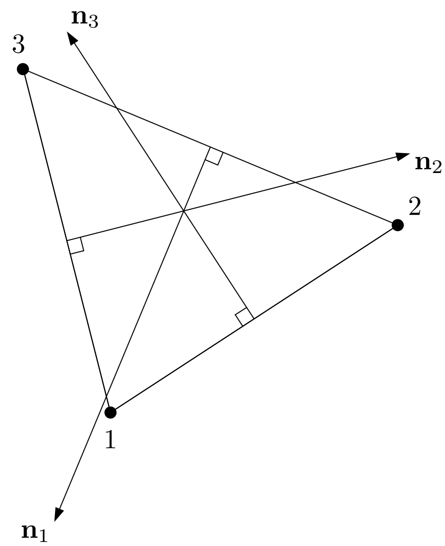

Figure 2: Triangular element and the scaled inward normals.

second-order upwind). The integration will still be exact for linear functions as long as the interpolation is exact for linear functions.

1.2 Quadrilaterals

For a general quadrilateral, we define the dual control volume around the node 1 by connecting the edge midpoints, ℓ and r, and the centroid, g (see Figure 3). The volume of the dual, V_{dual}, is related to the volume of the quadrilateral element, V, by

$$
\begin{aligned}
V_{dual} &= \frac{1}{8}V + \frac{1}{4}V_{T_1} && (13)\\
&= \frac{3}{8}V_{T_1} + \frac{1}{8}V_{T_3} && (14)\\
&= V_{T_1}\left[\frac{1}{2} + \frac{1}{8}\left(\frac{V_{T_3}}{V_{T_1}} - 1\right)\right], && (15)
\end{aligned}
$$

where V_{T_1} is the volume of the triangle, T_1, defined by nodes 1, 2, and 4; V_{T_3} is the volume of the triangle, T_3, defined by nodes 3, 4, and 2. If u is a linear function, the gradient is a global constant and can be expressed in the form of (4) for a triangle composed of any three of the nodes of the quadrilateral element. For our purpose, it is convenient to use the triangle T_1 to express the gradient:

$$
(\text{grad}\, u) = \frac{1}{2V_{T_1}} \sum_{i=1,2,4} u_i\, \mathbf{n}_i, \tag{16}
$$

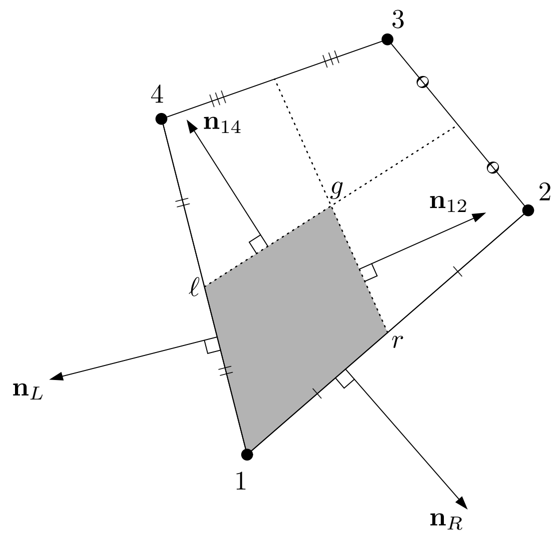

Figure 3: Quadrilateral element and a dual volume around the node 1.

where $\mathbf{n}_i$ are the scaled inward normals defined for the *triangle*, T_1. Now, we can evaluate the integral (1) as follows:

$$\int_{dual} \mathrm{grad}\, u\, dV = (\mathrm{grad}\, u) \times V_{dual}, \tag{17}$$

$$= (\mathrm{grad}\, u) \times V_{T_1} \left[\frac{1}{2} + \frac{1}{8} \left(\frac{V_{T_3}}{V_{T_1}} - 1 \right) \right], \tag{18}$$

$$= \left[\frac{1}{4} + \frac{1}{16} \left(\frac{V_{T_3}}{V_{T_1}} - 1 \right) \right] (u_1\, \mathbf{n}_1 + u_2\, \mathbf{n}_2 + u_4\, \mathbf{n}_4). \tag{19}$$

We then use the following geometric identities,

$$\mathbf{n}_1 = 2(\mathbf{n}_{12} + \mathbf{n}_{14}) + 4(\mathbf{n}_L + \mathbf{n}_R), \tag{20}$$

$$\mathbf{n}_2 = 2\left(\mathbf{n}_R + \mathbf{n}_{12} + \mathbf{n}_{14}\right), \tag{21}$$

$$\mathbf{n}_4 = 2\left(\mathbf{n}_L + \mathbf{n}_{12} + \mathbf{n}_{14}\right), \tag{22}$$

to write (19) as

$$\begin{aligned} \int_{dual} \mathrm{grad}\, u\, dV &= \frac{u_1 + u_2}{2}\mathbf{n}_{12} + \frac{u_1 + u_4}{2}\mathbf{n}_{14} + u_1 \mathbf{n}_L + u_1 \mathbf{n}_R \\ &+ \frac{1}{2}\left(\mathbf{n}_L + \mathbf{n}_{12}\right) u_4 + \frac{1}{2}\left(\mathbf{n}_R + \mathbf{n}_{14}\right) u_2 \\ &+ \frac{1}{16}\left(\frac{V_{T_3}}{V_{T_1}} - 1 \right) (u_1\, \mathbf{n}_1 + u_2\, \mathbf{n}_2 + u_4\, \mathbf{n}_4). \end{aligned} \tag{23}$$

The terms on the first line is a common edge-based finite-volume integration formula. Other terms account for a deviation from a parallelogram. They vanish if the element is a parallelogram (see Figure

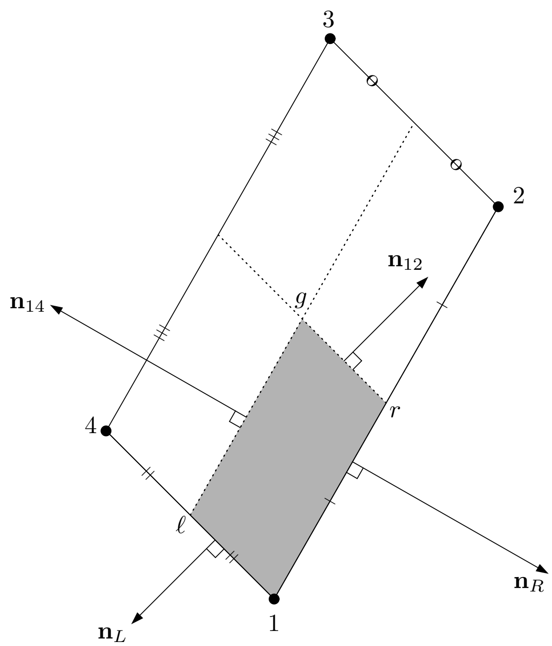

Figure 4: Parallelogram element and a dual volume around the node 1.

4):

$$\mathbf{n}_L + \mathbf{n}_{12} = 0, \tag{24}$$

$$\mathbf{n}_R + \mathbf{n}_{14} = 0, \tag{25}$$

$$\frac{V_{T_3}}{V_{T_1}} - 1 = 0. \tag{26}$$

Hence, the edge-based formula,

$$\int_{dual} \mathrm{grad}\, u\, dV = \frac{u_1 + u_2}{2}\mathbf{n}_{12} + \frac{u_1 + u_4}{2}\mathbf{n}_{14} + u_1\mathbf{n}_L + u_1\mathbf{n}_R, \tag{27}$$

is exact for linear functions if the element is a parallelogram.

1.3 Tetrahedra

For a tetrahedral element, we define the dual control volume around a node by connecting the node, the midpoints of three edges incident to the node, the centroids of three adjacent triangular faces, and the centroid of the tetrahedron. The volume of the dual, V_{dual}, is then precisely $\frac{1}{4}$ of the volume of the element, V:

$$V_{dual} = \frac{1}{4}V. \tag{28}$$

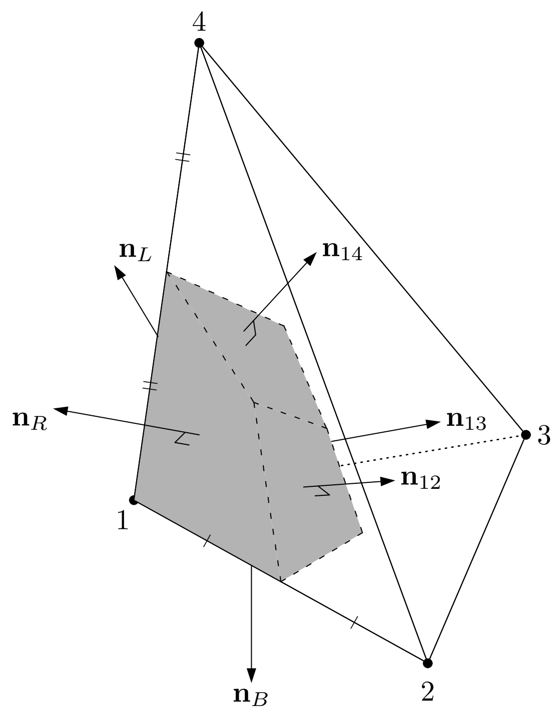

Figure 5: Tetrahedral element and a dual volume around the node 1.

If u is a linear function, the gradient is a global constant and can be uniquely expressed in terms of the nodal values over the element:

$$(\text{grad}\, u) = \frac{1}{3V} \sum_{i=1}^{4} u_i \, \mathbf{n}_i, \tag{29}$$

where u_i is the solution value at the node i and $\mathbf{n}_i$ is the scaled inward normal of the face opposite to the node i (i.e., the magnitude of $\mathbf{n}_i$ is equal to the face area). Then, we can evaluate the integral (1) as follows:

$$\int_{dual} \text{grad}\, u \, dV = (\text{grad}\, u) \times V_{dual}, \tag{30}$$

$$= (\text{grad}\, u) \times \frac{1}{4} V, \tag{31}$$

$$= \frac{1}{12} \sum_{i=1}^{4} u_i \, \mathbf{n}_i, \tag{32}$$

$$= \frac{1}{12} \left(u_1 \, \mathbf{n}_1 + u_2 \, \mathbf{n}_2 + u_3 \, \mathbf{n}_3 + u_4 \, \mathbf{n}_4 \right). \tag{33}$$

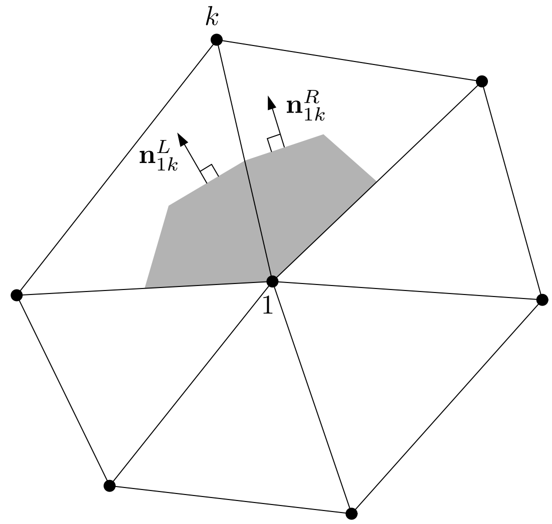

Figure 6: Triangular grid.

Substituting the following geometric identities,

$$\mathbf{n}_1 = 6(\mathbf{n}_{12} + \mathbf{n}_{13} + \mathbf{n}_{14}) + 9(\mathbf{n}_L + \mathbf{n}_R + \mathbf{n}_B), \tag{34}$$

$$\mathbf{n}_2 = 6\mathbf{n}_{12} + \frac{3}{2}\left(\mathbf{n}_R + \mathbf{n}_B\right), \tag{35}$$

$$\mathbf{n}_3 = 6\mathbf{n}_{13} + \frac{3}{2}\left(\mathbf{n}_B + \mathbf{n}_L\right), \tag{36}$$

$$\mathbf{n}_4 = 6\mathbf{n}_{14} + \frac{3}{2}\left(\mathbf{n}_L + \mathbf{n}_R\right), \tag{37}$$

we convert (33) into the edge-based finite-volume formula:

$$\begin{aligned}\int_{dual} \operatorname{grad} u \, dV &= \frac{u_1+u_2}{2}\mathbf{n}_{12} + \frac{u_1+u_3}{2}\mathbf{n}_{13} + \frac{u_1+u_4}{2}\mathbf{n}_{14} \\ &+ \left(\frac{6}{8}u_1 + \frac{1}{8}u_3 + \frac{1}{8}u_4\right)\mathbf{n}_L \\ &+ \left(\frac{6}{8}u_1 + \frac{1}{8}u_4 + \frac{1}{8}u_2\right)\mathbf{n}_R \\ &+ \left(\frac{6}{8}u_1 + \frac{1}{8}u_2 + \frac{1}{8}u_3\right)\mathbf{n}_B. \end{aligned} \tag{38}$$

This is exact for any linear function, u. It is really nice that the boundary integration uses the same weights for all boundary faces. Again, this allows us to perform the boundary integral without reference to the whole element data.

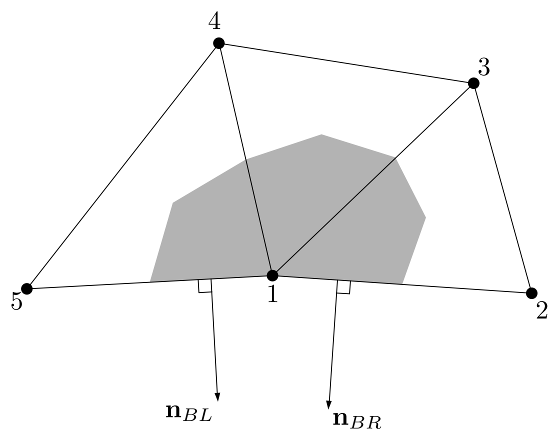

Figure 7: Triangular grid on a boundary.

2 Finite-Volume Integrations for a Group of Elements

2.1 Triangular Grids

For a triangular grid shown in Figure 6, we can apply the formula (12) for each triangle to construct a finite-volume integral over the dual control volume around j. Then, it is easy to see that all boundary contributions cancel out, and we obtain

$$\int_{dual} \mathrm{grad}\, u \, dV = \sum_k \frac{u_1 + u_k}{2} \left(\mathbf{n}_{1k}^L + \mathbf{n}_{1k}^R \right), \tag{39}$$

which is exact for a linear function, u. If the node 1 is on a boundary as in Figure 7, then we obtain, simply by collecting the contribution from each triangle,

$$\begin{aligned} \int_{dual} \mathrm{grad}\, u \, dV = \; & \frac{u_1 + u_2}{2} \mathbf{n}_{12} + \frac{u_1 + u_3}{2} \left(\mathbf{n}_{13}^L + \mathbf{n}_{13}^R \right) + \frac{u_1 + u_4}{2} \left(\mathbf{n}_{14}^L + \mathbf{n}_{14}^R \right) \\ & + \frac{u_1 + u_5}{2} \mathbf{n}_{15} + \left(\frac{5}{6} u_1 + \frac{1}{6} u_2 \right) \mathbf{n}_{BR} + \left(\frac{5}{6} u_1 + \frac{1}{6} u_5 \right) \mathbf{n}_{BL}. \end{aligned} \tag{40}$$

This is also exact for linear functions.

2.2 Quadrilateral Grids

It is easy to see from (23) that the finite-volume formula,

$$\int_{dual} \mathrm{grad}\, u \, dV = \sum_k \frac{u_1 + u_k}{2} \left(\mathbf{n}_{1k}^L + \mathbf{n}_{1k}^R \right), \tag{41}$$

for quadrilateral grids, is not exact for linear functions generally. If the quadrilaterals are parallelograms (or if those extra terms happen to add up to zero over quadrilaterals around a node), then this will be exact for linear functions.

2.3 Mixed Triangular/Quadrilateral Grids

If a stencil contains both triangles and quadrilaterals, then the finite-volume formula,

$$\int_{dual} \text{grad}\, u\, dV \quad = \quad \sum_k \frac{u_1 + u_k}{2} \left(\mathbf{n}_{1k}^L + \mathbf{n}_{1k}^R \right), \tag{42}$$

is not exact for linear functions generally because the boundary contributions from each single-element formula do not cancel each other along the edge between a triangle and a quadrilateral. However, if the intersection is *smooth*, e.g., a straight line, then it can be shown that the boundary contributions cancel overall, and the finite-volume formula will be exact for linear functions. Also note that even if the integration formula is not exact for linear functions, it will be exact for constant functions.

2.4 Tetrahedral Grids

The finite-volume formula is exact for linear functions for a general tetrahedral grid because the boundary contributions will cancel for all faces that shared by two adjacent tetrahedra. The finite-volume integral for a boundary node can also be exact for linear functions.

3 Green-Gauss Integration Forms

3.1 Triangles

Substituting the identities,

$$\mathbf{n}_1 \quad = \quad -\mathbf{n}_2 - \mathbf{n}_3, \tag{43}$$

$$\mathbf{n}_2 \quad = \quad -\mathbf{n}_3 - \mathbf{n}_1, \tag{44}$$

$$\mathbf{n}_3 \quad = \quad -\mathbf{n}_1 - \mathbf{n}_2, \tag{45}$$

into (8), we obtain

$$\int_{dual} \text{grad}\, u\, dV \quad = \quad \frac{1}{3} \left[\frac{u_2 + u_3}{2}(-\mathbf{n}_1) + \frac{u_3 + u_1}{2}(-\mathbf{n}_2) + \frac{u_1 + u_2}{2}(-\mathbf{n}_3) \right]. \tag{46}$$

This shows that the dual integral is $\frac{1}{3}$ of the Green-Gauss integral over the element boundary, ∂T:

$$\int_{dual} \text{grad}\, u\, dV \quad = \quad \frac{1}{3} \left(\oint_{\partial T} u\, \mathbf{n} dS \right)^{Green-Gauss}. \tag{47}$$

Naturally, this can also be expressed in terms of the Green-Gauss integral over the dual boundary. To see this, we substitute the identities,

$$\mathbf{n}_1 \quad = \quad \frac{5}{2}(\mathbf{n}_{12} + \mathbf{n}_{13}) + \frac{9}{2}(\mathbf{n}_L + \mathbf{n}_R), \tag{48}$$

$$\mathbf{n}_2 \quad = \quad \frac{5}{2}\mathbf{n}_{12} + \mathbf{n}_{13} + \frac{3}{2}\mathbf{n}_R, \tag{49}$$

$$\mathbf{n}_3 \quad = \quad \frac{5}{2}\mathbf{n}_{13} + \mathbf{n}_{12} + \frac{3}{2}\mathbf{n}_L, \tag{50}$$

into (8) to get

$$\begin{aligned}\int_{dual} \text{grad}\, u\, dV &= \frac{1}{2}\left(\frac{5}{6}u_1 + \frac{5}{6}u_2 + \frac{1}{3}u_3\right)\mathbf{n}_{12} + \frac{1}{2}\left(\frac{5}{6}u_1 + \frac{1}{3}u_2 + \frac{5}{6}u_3\right)\mathbf{n}_{13} \\ &+ \frac{3u_1+u_3}{4}\mathbf{n}_L + \frac{3u_1+u_2}{4}\mathbf{n}_R. \end{aligned} \quad (51)$$

This can be written as

$$\int_{dual} \text{grad}\, u\, dV = \frac{1}{2}(u_r+u_g)\,\mathbf{n}_{12} + \frac{1}{2}(u_g+u_\ell)\,\mathbf{n}_{13} + \frac{1}{2}(u_\ell+u_1)\,\mathbf{n}_L + \frac{1}{2}(u_1+u_r)\,\mathbf{n}_R, \quad (52)$$

where

$$u_r = \frac{u_1+u_2}{2}, \quad (53)$$

$$u_\ell = \frac{u_1+u_3}{2}, \quad (54)$$

$$u_g = \frac{u_1+u_2+u_3}{3}. \quad (55)$$

$$(56)$$

Hence, this is the Green-Gauss integration over the dual boundary:

$$\int_{dual} \text{grad}\, u\, dV = \left(\oint_{\partial(dual)} u\,\mathbf{n}dS\right)^{Green-Gauss}. \quad (57)$$

Therefore, the edge-based finite-volume integration formula (12) is equivalent to the Green-Gauss integration over the dual boundary as well as $\frac{1}{3}$ of the Green-Gauss integral over the element boundary, ∂T.

3.2 Tetrahedra

Substituting the identities,

$$\mathbf{n}_1 = -\mathbf{n}_2 - \mathbf{n}_3 - \mathbf{n}_4, \quad (58)$$

$$\mathbf{n}_2 = -\mathbf{n}_3 - \mathbf{n}_4 - \mathbf{n}_1, \quad (59)$$

$$\mathbf{n}_3 = -\mathbf{n}_4 - \mathbf{n}_1 - \mathbf{n}_2, \quad (60)$$

$$\mathbf{n}_4 = -\mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3, \quad (61)$$

into (33), we obtain

$$\begin{aligned}\int_{dual} \text{grad}\, u\, dV &= \frac{1}{4}\left[\frac{u_2+u_3+u_4}{3}(-\mathbf{n}_1) + \frac{u_3+u_4+u_1}{3}(-\mathbf{n}_2)\right. \\ &\left. + \frac{u_1+u_4+u_2}{3}(-\mathbf{n}_3) + \frac{u_1+u_2+u_3}{3}(-\mathbf{n}_4)\right]. \end{aligned} \quad (62)$$

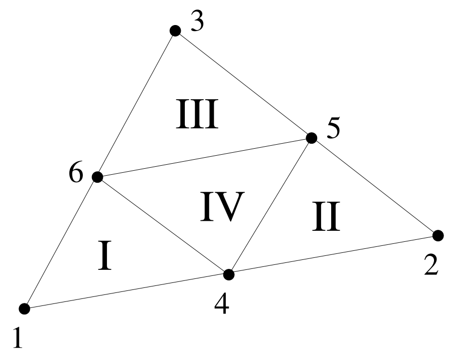

Figure 8: Quadratic triangular element.

This shows that the dual integral is $\frac{1}{4}$ of the Green-Gauss integral over the element boundary, ∂T:

$$\int_{dual} \operatorname{grad} u \, dV = \frac{1}{4} \left(\oint_{\partial T} u \, \mathbf{n} dS \right)^{Green-Gauss}. \tag{63}$$

Of course, it can be obtained also as the Green-Gauss integral over the dual boundary:

$$\int_{dual} \operatorname{grad} u \, dV = \left(\oint_{\partial(dual)} u \, \mathbf{n} dS \right)^{Green-Gauss}, \tag{64}$$

evaluated with the linearly interpolated nodal values at edge-midpoints and centroids. The proof is left as an exercise.

4 Quadrature Formulas

The integration formulas obtained in the previous sections can be used as quadrature rules. For example, for a triangular element, we write the edge-based formula (12) as

$$\int_{dual} \operatorname{grad} u \, dV = \hat{u}_\ell \mathbf{n}_{13} + \hat{u}_r \mathbf{n}_{12} + \left(\frac{5}{6} u_1 + \frac{1}{6} u_3 \right) \mathbf{n}_L + \left(\frac{5}{6} u_1 + \frac{1}{6} u_2 \right) \mathbf{n}_R, \tag{65}$$

where $\hat{u}_\ell$ and $\hat{u}_r$ denote function values at quadrature points: the midpoints of the edges 13 and 12 respectively (see Figure 1). As long as these values are evaluated such that they are exact for linear functions, the integration above remains exact for linear functions. This is nice. It allows us to introduce upwind-biased interpolation, for example. Alternatively, we can use other forms, (46) and (52), as quadrature formulas. With any linearly-exact function values, the integration will still be exact for linear functions. The same is true for other elements. In this viewpoint, the edge-based formulas are more efficient because they require less quadrature points than other formulas.

5 Remarks

In terms of integration formulas, simplices (triangle and tetrahedron) are very nice. Formulas are exact for linear functions for arbitrary grids. Quadrilaterals and hexahedron will also be nice (may be even nicer) if they are regular: parallellogram or parallelpiped.

It would be interesting to do the same analysis on high-order elements, such as a quadratic triangular element (a triangle with edge-midpoints added as additional degrees of freedom; see Figure 8): integrate (1) exactly, and arrange the result into an edge-based finite-volume formula.

For a quadratic element in Figure 8, since it has a self-similar structure, we might obtain a formula in the form of the Richardson extrapolation: the sum of $4/3$ of the linear triangle contribution and $-1/3$ of the sub-triangle contributions.

