On Positivity of the Galerkin Discretization

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1 Two Dimensions

In two dimensions, the Galerkin discretization can be written as (See Ref.[1])

$$\int_{\Omega} \Delta U d\Omega = -\frac{1}{4} \sum_{T \in \{T_j\}} \left(\frac{\mathbf{n}_j^T \cdot \mathbf{n}_j^T}{\Omega^T} \right) U_j - \frac{1}{4} \sum_{k \in \{k_j\}} \left(\frac{\mathbf{n}_k^L \cdot \mathbf{n}_j^L}{\Omega^L} + \frac{\mathbf{n}_k^R \cdot \mathbf{n}_j^R}{\Omega^R} \right) U_k, \tag{1}$$

where $\{k_j\}$ is a set of neighbors of j and the normals are scaled inward normals defined as in Figure 1. Using the fact that the inward normals sum up to zero within a triangle, we can rearrange this equation into

$$\int_{\Omega} \Delta U d\Omega = -\frac{1}{4} \sum_{k \in \{k_j\}} \left(\frac{\mathbf{n}_k^L \cdot \mathbf{n}_j^L}{\Omega^L} + \frac{\mathbf{n}_k^R \cdot \mathbf{n}_j^R}{\Omega^R} \right) (U_k - U_j), \tag{2}$$

which can be expressed also by the angles, θ^L and θ^R,

$$\int_{\Omega} \Delta U d\Omega = \frac{1}{2} \sum_{k \in \{k_j\}} \left(\cot\theta^L + \cot\theta^R \right) (U_k - U_j). \tag{3}$$

The discretization is positive if

$$\cot\theta^L + \cot\theta^R > 0, \tag{4}$$

for all edges, or equivalently if

$$\theta^L + \theta^R < \pi. \tag{5}$$

This is precisely the condition for the triangulation to be Delaunay [2]. Therefore, the Galerkin discretization is positive on the Delaunay triangulations.

Consider a triangular grid shown in Figure 2. For this stencil, the nodes 1 and 7 do not contribute to the discretization (2) because the angles θ^L and θ^R are both 90° and thus the coefficient $(\cot\theta^L + \cot\theta^R)$ vanishes. The Galerkin discretization (3) then becomes a standard 5-point finite-difference formula:

$$\int_{\Omega} \Delta U d\Omega = h^2 \left(\frac{U_4 - 2U_0 + U_8}{h^2} + \frac{U_6 - 2U_0 + U_2}{h^2} \right). \tag{6}$$

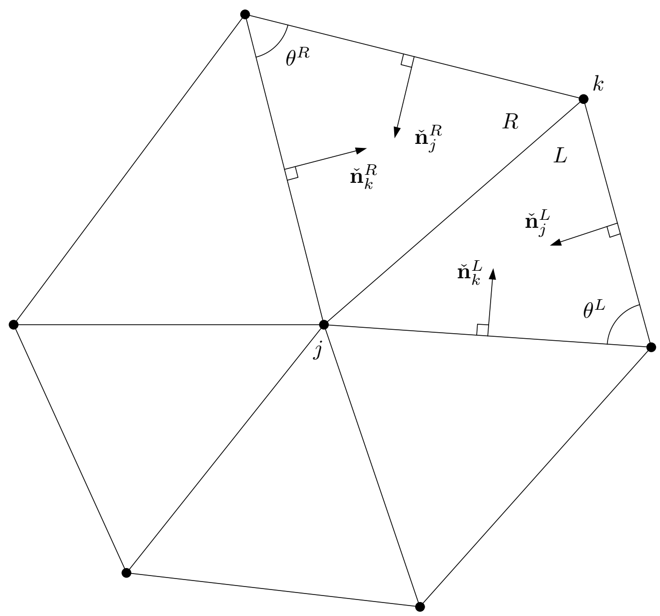

Figure 1: Node-centered stencil on triangular grids (the normals are not scaled).

Now, consider the perturbed stencil shown in Figure 3, which is obtained from the regular stencil by shifting the node j to the right. It is clearly seen that the angle between the edge 12 and $2j$ goes beyond $\pi/2$. Since the angle between the edges 18 and $8j$ remains $\pi/2$, the sum of the angles exceeds π, and therefore the coefficient $(\cot\theta^L + \cot\theta^R)$ for the node 1 becomes negative. It should be noted that this coefficient becomes negative not only for the perturbation in the positive x-direction, but also in the positive y-direction. Moreover, the same argument applies to the node 5: the coefficient becomes negative for any perturbation in the negative x- or y-direction. Therefore, the Galerkin discretization for this type of regular stencil will not be positive for arbitrary perturbation.

2 Three Dimensions

In three dimensions, the Galerkin discretization is given by

$$\begin{aligned}\int_{\Omega_j} \Delta U d\Omega &= \int_{\partial\Omega_j} \nabla U \cdot \mathbf{n} \\ &= -\sum_{T\in\{T_j\}} \frac{1}{9\Omega^T} \sum_{i\in\{i^T\}} U_i \left(\mathbf{n}_i \cdot \mathbf{n}_j^T\right) \\ &= -\frac{1}{9} \sum_{k\in\{k_j\}} \sum_{T\in\{T_{jk}\}} \left(\frac{\mathbf{n}_k^T \cdot \mathbf{n}_j^T}{\Omega^T}\right)(U_k - U_j) \qquad (7)\end{aligned}$$

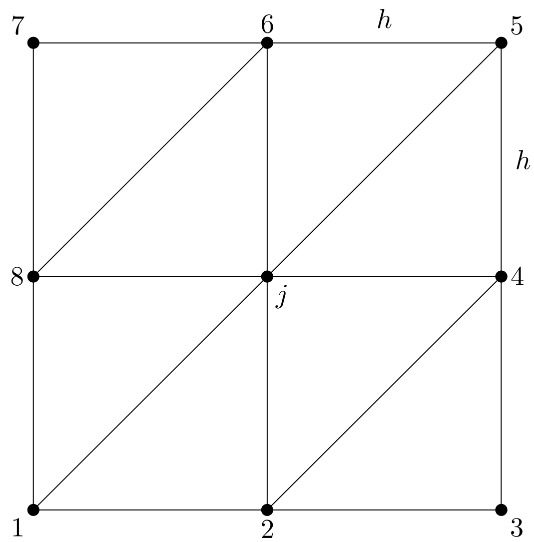

Figure 2: Regular stencil.

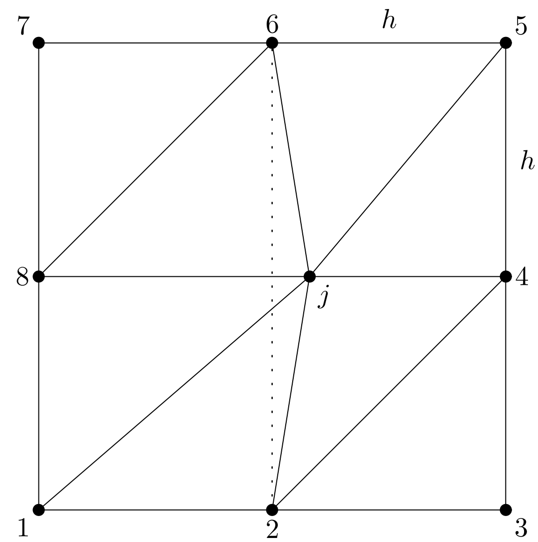

Figure 3: Perturbed stencil.

where $\{T_j\}$ denotes a set of tetrahedra that share a node j, Ω^T is the volume of the tetrahedron T, $\mathbf{n}_i$ is the inward directed area vector of the face opposite to the node i, $\mathbf{n}_j^T$ is the scaled inward directed area vector opposite to the node j in the cell T. Also, $\{T_{jk}\}$ denotes a set of tetrahedra that share the edge jk. The discretization can be written also as

$$\int_\Omega \Delta U d\Omega \quad = \quad \frac{2}{3} \sum_{k \in \{k_j\}} \sum_{T \in \{T_{jk}\}} \ell_T \cot\theta_T (U_k - U_j), \tag{8}$$

where ℓ_T is the length of the edge shared by the faces associated with $\mathbf{n}_k^T$ and $\mathbf{n}_j^T$, and θ_T is the angle between the two faces. Hence, the condition for positivity is

$$\sum_{T \in \{T_{jk}\}} \ell_T \cot\theta_T \geq 0, \quad \forall k \in \{k_j\}. \tag{9}$$

For simplicity, let us consider a cube of size h divided into 6 tetrahedra, and take two nodes connected by the diagonal edge to be j and k (this edge is surrounded by all the six tetrahedra). In this case, $\ell_T = h$ for all, and the positivity condition is given by

$$\sum_{T \in \{T_{jk}\}} \cot\theta_T \geq 0. \tag{10}$$

But the angle θ_T is precisely equal to $\pi/2$ for all tetrahedra, and therefore the coefficient vanishes. Now, consider perturbing the node j in any outward direction. It can be easily shown that it only makes some of the angles larger than $\pi/2$, and the rest of the angles remain $\pi/2$ (similar to what happens to the quadrilateral $12j8$ in Figure 3). This means that non-zero contributions to the coefficient are all negative.

Consequently, the Galerkin discretization for a perturbed regular tetrahedral stencil is not positive. This conclusion was confirmed numerically. For regular tetrahedral grids in a cube with extremely small nodal perturbations (order of $h/10000$), *all* nodes were found to have negative coefficients although very small in magnitude.

3 Remarks

It seems very likely in more general cases that negative coefficients arise at many (or maybe all) nodes in the grid. Ignoring them all, we would end up with inconsistent discretization almost everyewhere.

References

[1] B. Diskin, J. L. Thomas, E. J. Nielsen, H. Nishikawa, J. A. White, Comparison of node-centered and cell-centered unstructured finite-volume discretizations: Viscous fluxes, AIAA Journal 48 (7) (2010) 1326–1338.

[2] T. J. Barth, Numerical aspects of computing viscous high reynolds number flows on unstructured meshes, AIAA Paper 91-0721, 1991.

