Edge-Based Form of Galerkin Source Term Discretization

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Abstract

We here consider Galerkin discretization of a source term. Löhner mentions in Ref.[1] (in a slightly different context) that it can be implemented in a loop over edges, but details are not shown. I suppose that the following is what he meant.

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1 Two Dimensions

Consider a triangulation defined by the set $\{I\}$ of nodes and the set $\{T\}$ of triangles. Given solution values at nodes, we consider discretizing a dual-control-volume integral around node i of a source term, $s(x,y)$,

$$\int_{dual_i} s(x,y)\,dxdy, \tag{1}$$

by the P_1 Galerkin method. Let s_h be a piecewise linear representation of s:

$$s_h = \sum_{i\in\{I\}} s_i\,\phi_i(x,y), \tag{2}$$

where $s_i = s(x_i, y_i)$ and $\phi_i(x,y)$ denotes a piecewise linear basis function that takes 1 at node i and 0 at all other nodes. The Galerkin discretization of Equation (1) is

$$\int_{dual_i} s(x,y)\,dxdy \approx \int_{\Omega} \phi_i\, s_h\, dxdy \tag{3}$$

where Ω denotes the entire domain. Due to the compactness of ϕ_i, it suffices to perform the integration over the set $\{T_i\}$ of triangles that share the node i (see Figure 1). The result is

$$\int_{dual_i} s(x,y)\,dxdy \approx \frac{1}{6}\sum_{T\in\{T_i\}}\left(s_i + \frac{1}{2}\,s_{i_\ell} + \frac{1}{2}\,s_{i_r}\right)V_T, \tag{4}$$

where i_ℓ and i_r denote the two nodes of the triangle T other than i, V_T denotes the volume of the triangle T. This formula can be written as a sum over the neighbors:

$$\int_{dual_i} s(x,y)\,dxdy \approx \frac{1}{6}\sum_{k\in\{K_i\}}\frac{s_i+s_k}{2}\,(V_L+V_R)\,, \tag{5}$$

where $\{K_i\}$ denotes a set of neighbor nodes of i, and V_L and V_R are the volumes of the elements on the left and the right of the edge, respectively (see Figure 2). It can be implemented in an edge-loop provided the edge-based data have access to the volumes of the adjacent elements.

Lumped Formula:

If the right hand side is lumped (i.e., $s_{i_\ell} = s_{i_r} = s_i$), the Galerkin discretization (4) becomes

$$\int_{dual_i} s(x,y)\,dxdy \approx \frac{1}{3}\left(\sum_{T\in\{T_i\}} V_T\right)s_i = s_i V_{dual_i}, \tag{6}$$

which is a typical one-point quadrature.

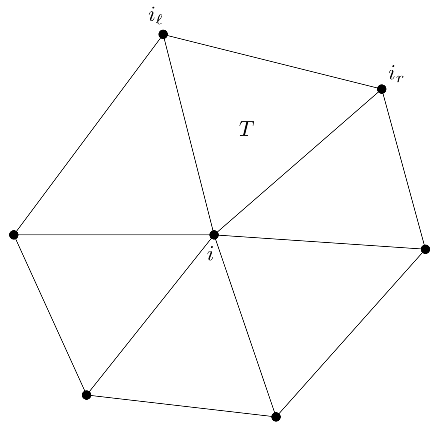

Figure 1: A set $\{T_i\}$ of triangles sharing the node i.

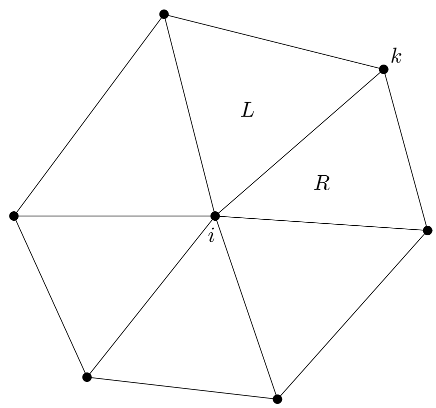

Figure 2: Edge defined by nodes, i and k, and the adjacent left and right elements, L and R.

2 Three Dimensions

For a tetrahedral grid defined by the set of tetrahedra, $\{T\}$, we obtain

$$\int_{dual_i} s(x,y,z)\, dxdydz \approx \int_{\Omega} \phi_i\, s_h\, dxdydz = \sum_{T\in\{T_i\}} \left(\frac{1}{10} s_i + \frac{1}{20}\, s_{i_1} + \frac{1}{20}\, s_{i_2} + \frac{1}{20}\, s_{i_3} \right) V_T, \tag{7}$$

where ϕ_i is a 3D version of the linear basis function at node i, $\{T_i\}$ denotes the set of tetrahedra sharing the node i, V_T denotes the volume of the tetrahedron T, and i_1, i_2, and i_3 denote the three nodes of the tetrahedron T other than i. This formula can be written as a sum over the neighbors:

$$\int_{dual_i} s(x,y,z)\, dxdydz \approx \frac{1}{20} \sum_{k\in\{K_i\}} \left[\left(\frac{2}{3} s_i + s_k \right) \sum_{T\in\{T_k\}} V_T \right], \tag{8}$$

where $\{K_i\}$ denotes a set of neighbor nodes of i, $\{T_k\}$ denotes the set of tetrahedra sharing the edge $\{i,k\}$, and V_L and V_R are the volumes of the tetrahedra on the left and the right of the edge, respectively. As in two dimentions, the formula reduces to a typical one-point quadrature if we set $s_{i_1} = s_{i_2} = s_{i_3} = s_i$.

References

[1] R. Löhner. CFD via unstructured grids: Trends and applications. In *Frontiers of Computational Fluid Dynamics*, pages 117–133. John Wiley & Son Ltd., 1994.