A short version has been published as [AIAA Journal, 2010 v.48, no.7] (See Appendix).

Galerkin Discretization on Triangular Grids

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1 Galerkin Discretization on Arbitrary Stencil (Triangles)

The standard Galerkin discretization of the Laplace equation ($\nabla^2 u = 0$) over a triangular mesh results in the followng equation at node j:

$$-\int\int_\Omega \nabla^2 u_h = \int\int_\Omega \nabla u_h \cdot \nabla \phi_j = 0, \tag{1}$$

where u_h is a linear approximation of the solution u and ϕ_j is the basis function at the node j. The gradient of the linear basis funciton ϕ_j is constant over each triangle:

$$\nabla \phi_j^T = \frac{\mathbf{n}_j^T}{2S_T}, \tag{2}$$

where S_T is the area of the triangle T, and $\mathbf{n}_j^T$ is the scaled inward normal vector opposite to the node j (the magnitude is equal to the length of the edge opposite to the node j). The linear approximation, u_h, is given by

$$u_h = \sum_{i \in \{nodes\}} u_i \phi_i. \tag{3}$$

The gradient of u_h is also constant over each triangle:

$$\nabla u^T = \frac{1}{2S_T} \sum_{i \in \{j_T\}} u_i \mathbf{n}_i^T, \tag{4}$$

where j_T is a set of three vertices of the triangle T and u_i is the nodal value of u at the node i.

Because all the gradients are constant over each triangle and ϕ_j is zero outside a set of triangles, $\{T_j\}$, that share the node j (including the boundary), the Galerkin discretization (1) becomes (simply substitute the constant gradients)

$$\sum_{T \in \{T_j\}} \frac{1}{4S_T} \sum_{i \in \{j_T\}} u_i \left(\mathbf{n}_i^T \cdot \mathbf{n}_j^T \right) = 0. \tag{5}$$

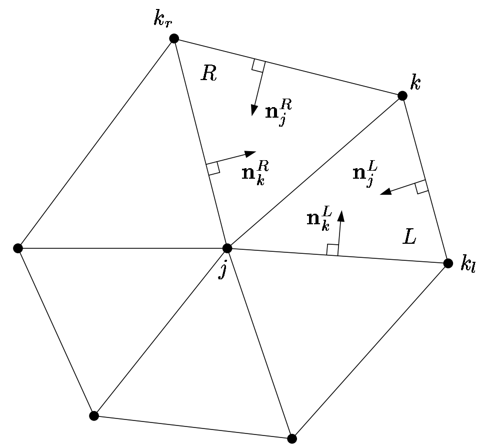

Figure 1: Stencil

This can be rearranged into the sum over nodes rather than triangles around j,

$$\frac{1}{4}\sum_{T\in\{T_j\}}\left(\frac{\mathbf{n}_j^T\cdot\mathbf{n}_j^T}{S_T}\right)u_j+\frac{1}{4}\sum_{k\in\{k_j\}}\left(\frac{\mathbf{n}_k^L\cdot\mathbf{n}_j^L}{S_T}+\frac{\mathbf{n}_k^R\cdot\mathbf{n}_j^R}{S_T}\right)u_k=0, \tag{6}$$

where $\{k_j\}$ is a set of nodes that are directly connected to j. See Figure 1 for the definitions of the normals. Note that the coefficients for u_k can be expressed in terms of edge angles also (the product of the edge lengths times the cotangent of the angle inbetween).

2 Special Cases

2.1 Regular Uniform Grid

On a regular stencil shown in Figure 2, the Galerkin discretization (6) simplifies to

$$4u_j-u_2-u_4-u_6-u_8=0, \tag{7}$$

or

$$-\frac{u_4-2u_j+u_8}{h^2}-\frac{u_2-2u_j+u_6}{h^2}=0, \tag{8}$$

which corresponds to the standard 5-point finite-difference discretization. Note in particular that there are no contributions from nodes 1 and 5 because the dot products of the normals in their coefficients vanish (the normals are perpendicular to each other).

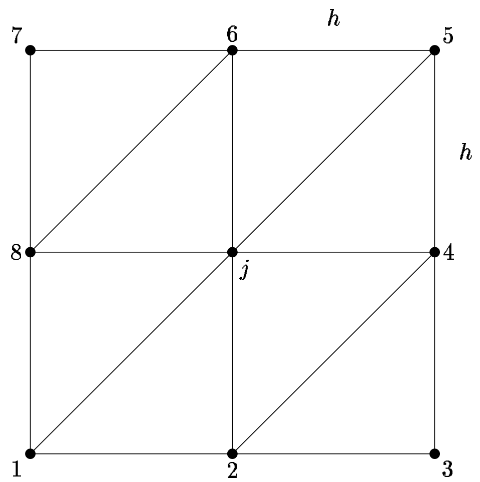

Figure 2: Regular triangular grid

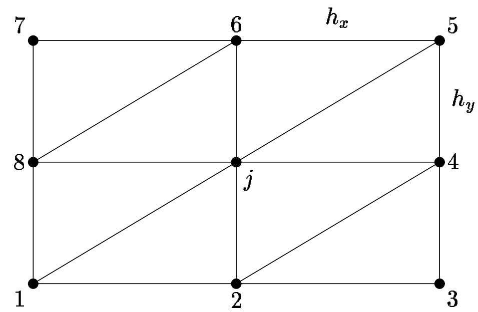

Figure 3: Regular non-uniform triangular grid

2.2 Regular High Aspect Ratio Grid

Next, consider a high aspect ratio grid shown in Figure 3. For this stencil, the Galerkin discretization (6) simplifies to

$$\frac{2(h_x^2+h_y^2)}{h_x h_y}u_j - \frac{h_x}{h_y}u_2 - \frac{h_x}{h_y}u_6 - \frac{h_y}{h_x}u_4 - \frac{h_y}{h_x}u_8 = 0, \tag{9}$$

which is

$$\frac{2h_x}{h_y}u_j + \frac{2h_y}{h_x}u_j - \frac{h_x}{h_y}(u_2+u_6) - \frac{h_y}{h_x}(u_4+u_8) = 0, \tag{10}$$

and therefore, we have

$$-\frac{u_4-2u_j+u_8}{h_x^2} - \frac{u_2-2u_j+u_6}{h_y^2} = 0. \tag{11}$$

This is again nothing but the standard finite-difference discretization. Note that there are no contributions from nodes 1 and 5 again.

2.3 Regular High Aspect Ratio Grid with Various Diagonal Splittings

It is easy to show that the results in the previous subsections are independent of diagonal connections. Any diagonal swappings do not change the formulas (8) and (11). To see this, consider the coefficient for the solution u_2:

$$\frac{1}{4}\left(\frac{\mathbf{n}_2^L \cdot \mathbf{n}_j^L}{S_L} + \frac{\mathbf{n}_2^R \cdot \mathbf{n}_j^R}{S_R}.\right) \tag{12}$$

There are four possible diagonal splittings as shown in Figures 4 to 7. It is obvious that the dot products in the coefficient, such as $\mathbf{n}_2^L \cdot \mathbf{n}_j^L$, are all equal to $-h_x^2$ for all diagonal splittings. Also noting that $S_T = h_x h_y/2$ for all triangles, we find

$$\frac{1}{4}\left(\frac{\mathbf{n}_2^L \cdot \mathbf{n}_j^L}{S_L} + \frac{\mathbf{n}_2^R \cdot \mathbf{n}_j^R}{S_R}\right) = \frac{1}{4}\left(\frac{-h_x^2}{h_x h_y/2} + \frac{-h_x^2}{h_x h_y/2}\right) = -\frac{h_x}{h_y}, \tag{13}$$

for *all possible diagonal splittings*. The same is true for u_6. In the same way, we find

$$\frac{1}{4}\left(\frac{\mathbf{n}_4^L \cdot \mathbf{n}_j^L}{S_L} + \frac{\mathbf{n}_4^R \cdot \mathbf{n}_j^R}{S_R}\right) = -\frac{h_y}{h_x}, \tag{14}$$

$$\frac{1}{4}\left(\frac{\mathbf{n}_8^L \cdot \mathbf{n}_j^L}{S_L} + \frac{\mathbf{n}_8^R \cdot \mathbf{n}_j^R}{S_R}\right) = -\frac{h_y}{h_x}, \tag{15}$$

for u_2 and u_8. Also, for u_j, we find

$$\frac{1}{4}\sum_{T\in\{T_j\}}\left(\frac{\mathbf{n}_j^T \cdot \mathbf{n}_j^T}{S_T}\right) = \frac{2(h_x^2 + h_y^2)}{h_x h_y}, \tag{16}$$

for *all possible diagonal splittings*. Therefore, the Galerkin discretization (6) reduces to

$$\frac{2(h_x^2 + h_y^2)}{h_x h_y}u_j - \frac{h_x}{h_y}u_2 - \frac{h_x}{h_y}u_6 - \frac{h_y}{h_x}u_4 - \frac{h_y}{h_x}u_8 = 0, \tag{17}$$

and thus

$$-\frac{u_4 - 2u_j + u_8}{h_x^2} - \frac{u_2 - 2u_j + u_6}{h_y^2} = 0, \tag{18}$$

for *all possible diagonal splittings*.

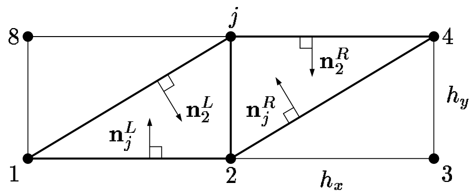

Figure 4: Stencil A for the coefficient of u_2.

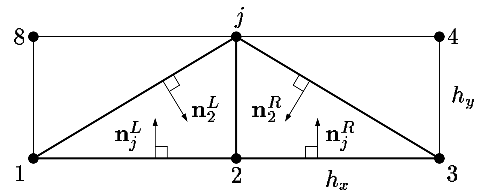

Figure 5: Stencil B for the coefficient of u_2.

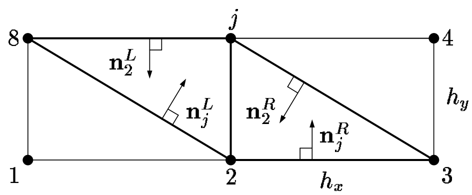

Figure 6: Stencil C for the coefficient of u_2.

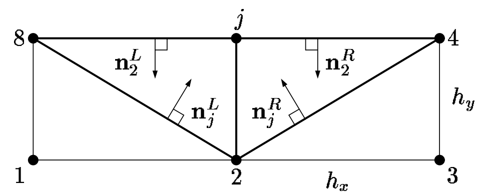

Figure 7: Stencil D for the coefficient of u_2.

2.4 Stretched (Non-Uniform) Grid

Consider the same high aspect ratio grid with non-uniform spacing in y direction (see Figure 8). In this case, the previous results are still valid at nodes 2 and 6. But h_y' appears instead of h_y in the coefficient of u_2.

$$\frac{1}{4}\left(\frac{\mathbf{n}_2^L\cdot\mathbf{n}_j^L}{S_L}+\frac{\mathbf{n}_2^R\cdot\mathbf{n}_j^R}{S_R}\right)=-\frac{h_x}{h_y'}, \tag{19}$$

$$\frac{1}{4}\left(\frac{\mathbf{n}_6^L\cdot\mathbf{n}_j^L}{S_L}+\frac{\mathbf{n}_6^R\cdot\mathbf{n}_j^R}{S_R}\right)=-\frac{h_x}{h_y}. \tag{20}$$

For the node 4, the associated left and right triangles now have different areas. Taking into account this, we obtain

$$\frac{1}{4}\left(\frac{\mathbf{n}_4^L\cdot\mathbf{n}_j^L}{S_L}+\frac{\mathbf{n}_4^R\cdot\mathbf{n}_j^R}{S_R}\right)=\frac{1}{4}\left(\frac{-{h_y'}^2}{h_x h_y'/2}+\frac{-{h_y}^2}{h_x h_y/2}\right)=-\frac{1}{2}\left(\frac{h_y'}{h_x}+\frac{h_y}{h_x}\right). \tag{21}$$

Similarly, we obtain, for the node 8,

$$\frac{1}{4}\left(\frac{\mathbf{n}_8^L\cdot\mathbf{n}_j^L}{S_L}+\frac{\mathbf{n}_8^R\cdot\mathbf{n}_j^R}{S_R}\right)=-\frac{1}{2}\left(\frac{h_y'}{h_x}+\frac{h_y}{h_x}\right). \tag{22}$$

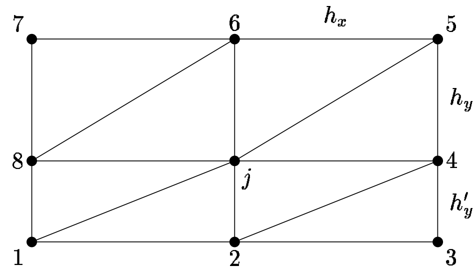

Figure 8: Stencil

For the node j, we obtain

$$\frac{1}{4}\sum_{T\in\{T_j\}}\left(\frac{\mathbf{n}_j^T\cdot\mathbf{n}_j^T}{S_T}\right)=\frac{h_x^2+h_y^2}{h_xh_y}+\frac{h_x^2+{h'_y}^2}{h_xh'_y}. \tag{23}$$

We emphasize that these results are independent of diagonal splittings. Finally, the Galerkin discretization (6) becomes

$$\left(\frac{h_x^2+h_y^2}{h_xh_y}+\frac{h_x^2+{h'_y}^2}{h_xh'_y}\right)u_j-\frac{h_x}{h'_y}u_2-\frac{h_x}{h_y}u_6-\frac{1}{2}\left(\frac{h'_y}{h_x}+\frac{h_y}{h_x}\right)u_4-\frac{1}{2}\left(\frac{h'_y}{h_x}+\frac{h_y}{h_x}\right)u_8=0. \tag{24}$$

This can be rearranged into the following,

$$-\frac{u_4-2u_j+u_8}{h_x^2}-\frac{\frac{u_6-u_j}{h_y}-\frac{u_j-u_2}{h'_y}}{\frac{h_y+h'_y}{2}}=0. \tag{25}$$

The second term approximates $\frac{\partial^2 u}{\partial y^2}$. It is second-order accurate not at the node j but somewhere off above it. Therefore, this formula is not precisely second-order accurate. In fact, inserting a smooth function expanded around j into the formula (25), we find,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{3}\frac{\partial^3 u}{\partial y^3}(h_y-h'_y)+O(h^2)=0. \tag{26}$$

This shows that the formula is in general first-order accurate. But the actual solution error (discretization error) could still be second-order accurate.

We emphasize again that all results above are true for an arbitrary diagonal splitting.

