A Geometric Interpretation of Fluctuations

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Abstract

In this paper, we describe a derivation of the fluctuations of the differential equations in terms of their geometrical properties. A solution of a set of differential equations, with $(n-p)$ dependent variables and p independent variables, can be regarded as forming a p-dimensional submanifold in n-dimensional manifold. Such a submanifold is called an integral manifold. In the theory of exterior differential forms, the integral manifold is defined as a manifold whose tangent vectors annul, at every point on itself, the differential forms equivalent to the original differrential equations. Following this particular concept, we shall derive fluctuations for three example problems; one-, two- and three-dimensional problems, and discuss their geometric interpretation in each dimension.

1 Introduction

The fluctuation splitting scheme for triangular grids has been extended in [1] by including nodal coordinates as additional unknowns, which allows movements of nodes during the computation. This simple extention has revealed some notable features of the numerical scheme. It has been found that the fluctuation is a discrete version of a partial differential equation as well as its hodograph equation, i.e. the fluctuation expresses two set of equaitons in a single form. Therefore treating all variables as unknowns, we are in effect solving both equations simultaneously; one being responsible for solutions and the other being responsible for a grid. The advantage of such an approach is the automatic grid adaptation that faithfully responds to the physics of governing equations.

Another, perhaps more intriguing, interpretation of the numerical sheme is that it attempts to approximate a solution submanifold[1]. More precisely it attempts to find a location of each element in a triangulation such that the elements fit together to form a surface approximating a solution submanifold. This viewpoint is probably more general and gives a clear picture of what the numerical scheme tries to do. In order to assert this however we need to verify that the fluctuations really represent a geometrical property of a solution submanifold, and clarify such a property. We shall do this in this paper by using differential forms that perfectly fit the situation. A geometric theory of partial differential equations were developed by Cartan[2] in the language of differrential forms. The theory deals with a set of differential forms which is equivalent to a given system of differential equations. A solution to the set of forms are then understood as a submanifold (or integral manifold) whose tangent vectors annul the forms everywhere on it, in contrast with a solution to differential equations as a function of independent variables. See [2][3][4][5] for details. We shall consider a numerical approximation to a solution manifold, typically by triangulation. A discrete version of a set of differential forms are obtained by requiring them to vanish when approximate tangent vectors are contracted with the set of forms. We shall show that the resulting expressions are identical to fluctuations, thus proving the geometric interpretation of fluctuations.

We shall describe the details of the derivation by examples. In the next section we derive fluctuations for a one-dimensional problem which is so simple that one can easily develop a concrete picture of its geometry. And then a two-dimensional example follows whose geometrical picture is also clearly understood. Finally a three-dimensional example is given.

2 Friedrichs' model

Friedrichs' model of the boundary-layer momentum equation is the second-order differential equation,

$$\epsilon \frac{d^2u}{dy^2} + \frac{du}{dy} - a = 0 \tag{1}$$

with

$$u(0) = 0, \quad u(1) = 1 \tag{2}$$

where ϵ and $a(< 1)$ model the reciprocal of the Reynolds number and the x-derivative term respectively. In order to write this in terms of differential forms, we first reduce the equations into the first-order set by introducing $J = \frac{du}{dy}$ as an additional unknown. We thus have

$$\epsilon \frac{dJ}{dy} + \frac{du}{dy} - a = 0 \tag{3}$$

$$\frac{du}{dy} - J = 0 \quad , \tag{4}$$

rewritten as

$$\epsilon dJ + du - a\, dy = 0 \tag{5}$$

$$du - J\, dy = 0 \tag{6}$$

On a three-dimensional manifold with the coordinates (u, J, y), we define the two one-forms ϕ and ψ as

$$\phi = \epsilon\, \tilde{d}J + \tilde{d}u - a\, \tilde{d}y \tag{7}$$

$$\psi = \tilde{d}u - J\, \tilde{d}y \tag{8}$$

where $\tilde{d}u$, $\tilde{d}J$ and $\tilde{d}y$ are now one-forms. These are the differential forms equivalent to the original differential equations, which may be verified by sectioning the forms into a solution manifold. The sectioning is done by substituting $\tilde{d}u = u_y \tilde{d}y$ and $\tilde{d}J = J_y \tilde{d}y$ into the above equations and then requiring the components of the forms to vanish[1]. This is also equivalent to imposing the independence of y. In order to assert the equivalence to

[1] The solution submanifold is represented by $(u(y), J(y))$ satisfying (3)(4) with y as a parameter.

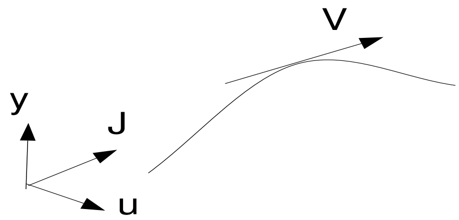

Figure 1: A solution curve and a tangent vector.

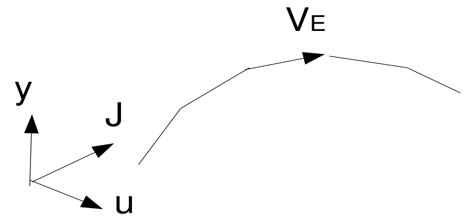

Figure 2: Approximatation curve and a tangent vector.

(3) and (4) however it still remains to show that the set of forms is closed. A closed set is defined as a set whose elements generate a complete ideal[2] and the exterior derivatives of the elements are also in the ideal. It is easy to show that a set of forms $\{\beta_i\}$ is closed if exterior derivatives of the forms can be written as a linear combination of $\{\beta_i\}$, i.e. $\sum \gamma_i \wedge \beta_i$ where γ_i is any form. Therefore in order to show that a set of forms is closed, we need to show that the exterior derivatives of the forms can be expressed as a linear combination of the forms themselves. In our case, we find

$$\tilde{d}\phi = 0 \tag{9}$$

$$\tilde{d}\psi = \frac{1}{\epsilon}\tilde{d}y \wedge \phi - \frac{1}{\epsilon}\tilde{d}y \wedge \psi \quad . \tag{10}$$

Therefore they in fact constitute a closed ideal of differential forms. Cartan's theory[2] then guarantees the existence of the solution submanifold of the forms, i.e. any local surface element that annul ϕ and ψ will fit together to form a solution manifold. It is interesting to note that equation (9) implies that ϕ is closed. In fact it is also exact since we can write

$$\phi = \tilde{d}(\epsilon\, J + u - a\, y) \quad . \tag{11}$$

It states that the quantity $\epsilon\, J + u - a\, y$ is conserved along the solution manifold. In the language of the exterior differential forms, solving a system of differential equations is equivalent to finding a submanifold whose tangent vectors annul the forms, which we write

$$\phi(V) = \epsilon\, \tilde{d}J(V) + \tilde{d}u(V) - a\, \tilde{d}y(V) = 0 \tag{12}$$

$$\psi(V) = \tilde{d}u(V) - J\, \tilde{d}y(V) = 0 \tag{13}$$

where the tangent vector is denoted by V. An analytical method to solve the problem is discussed in [2][3][4][5]. Here we shall discuss a discrete approximation to the solution.

In general, for a problem with $(n-p)$ dependent variables and p independent variables its integral manifold is p-dimensional because there are only p variables which can freely chosen or because there are $(n-p)$ restrictions on n-dimensional manifold. Hence the integral manifold of the Friedrichs' model is a one-dimensional curve in three-dimensional manifold (Figure 1). We will approximate the solution curve by a curve which consists of a set of linear elements $\{E\}$ connecting its elements to join the two boundary vertices

[2]A complete ideal is all the forms at a point whose restriction to a tangent space vanishes.

(0,J,0) and (1,J,1) with its orientation as positive in the direction from (0,J,0) to the other (see Figure 2). The unknowns in the discrete problem are then the coordinates of all the vertices, i.e. (u, J, y), that define a particular curve. On the approximate curve we have a finite number of tangent vectors each of which is pointing from one vertex to the next vertex in the positive direction, i.e. the element itself. We then define the tangent vector V_E as

$$V_E = (\Delta u_E, \Delta J_E, \Delta y_E)^T \tag{14}$$

where Δu_E denotes the projected length of the element E onto u-axis, and similarly for the others. It is important to remember here that differential forms are defined at a point on a manifold. In this example we will assume that the differential forms and the tangent vectors are defined at the mid-point of the element. This leads to the following equations for element $E \in \{E\}$.

$$\phi(V_E) = \epsilon\, \tilde{d}J(V_E) + \tilde{d}u(V_E) - a\, \tilde{d}y(V_E) \tag{15}$$

$$\psi(V_E) = \tilde{d}u(V_E) - \bar{J}_E\, \tilde{d}y(V_E) \tag{16}$$

where $\bar{J}_E$ denotes a value of J at the mid-point of the edge E. Finally we get

$$\Phi_E \equiv \phi(V_E) = \epsilon \Delta J_E + \Delta u_E - a\, \Delta y_E \tag{17}$$

$$\Psi_E \equiv \psi(V_E) = \Delta u_E - \bar{J}_E\, \Delta y_E \quad . \tag{18}$$

We may call these quantities fluctuations since these expressions coincide with those obtained by integrating the equations (3) and (4) over a small interval Δy on y-axis assuming that u and J vary linearly within the interval. Therefore, conversely, the fluctuations can be thought of as the geometrical property of the solution curve, i.e. numerical values of the fluctuations are the errors in aligning the tangent vectors along the solution curve.

3 Cauchy-Riemnann equations

In this section we consider the Cauchy-Riemann system given by

$$u_x + v_y = 0 \tag{19}$$

$$v_x - u_y = 0 \tag{20}$$

where u and v are the velocity components in an incompressible and irrotational flow. We define the two-forms ϕ and ψ as

$$\phi = \frac{1}{2}\tilde{d}u \wedge \tilde{d}y - \frac{1}{2}\tilde{d}v \wedge \tilde{d}x \tag{21}$$

$$\psi = \frac{1}{2}\tilde{d}v \wedge \tilde{d}y + \frac{1}{2}\tilde{d}u \wedge \tilde{d}x \quad . \tag{22}$$

The equivalence to the original equations may be verified by imposing the independence of x and y. On substituting $\tilde{d}u = u_x\, \tilde{d}x + u_y\, \tilde{d}y$ and $\tilde{d}v = v_x\, \tilde{d}x + v_y\, \tilde{d}y$, we have

$$\phi = \frac{1}{2}(u_x + v_y)\, \tilde{d}x \wedge \tilde{d}y \tag{23}$$

$$\psi = \frac{1}{2}(v_x - u_y)\, \tilde{d}x \wedge \tilde{d}y \tag{24}$$

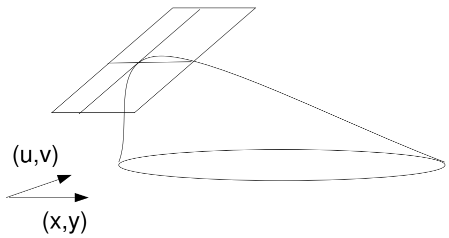

Figure 3: A solution surface and a tangent space in the 4-dimensional space.

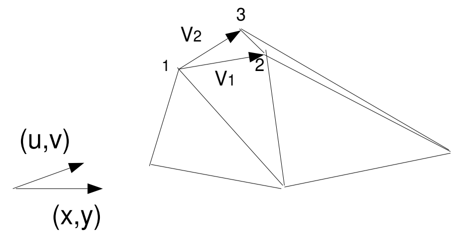

Figure 4: A triangulation and tangent vectors of a triangle $T \in \{T\}$.

and retrieve the original equations by requiring the coefficients to vanish. Again it is obvious that the forms constitute a closed ideal. It is interesting to note that the so-called hodograph equations are obtainable by substituting instead $\tilde{d}x = x_u\,\tilde{d}u + x_v\,\tilde{d}v$ and $\tilde{d}y = y_u\,\tilde{d}u + y_v\,\tilde{d}v$ into the two-forms.

The problem now becomes finding a two-dimensional submanifold, i.e. a surface, whose tangent vectors annul the two-forms. Note that such tangent vectors form a vector space at a point on the integral surface, whose elements are tangent to the surface at the point (see Figure 3). In the present case the integral manifold is a two-dimensional submanifold in a four-dimensional manifold whose coordinates are (u, v, x, y). We will then construct an approximate submanifold by triangulation (see Figure 4). The outer boundary of the surface is defined by given boundary conditions and the interior domain is divided by a set of triangles $\{T\}$. Consider a typical triangular element $T \in \{T\}$ which is defined by its vertices 1, 2 and 3. The tangent space being two-dimentional, we may define the two tangent vectors as the two of the three sides of the triangle,

$$V_1 = (\Delta u_2, \Delta v_2, \Delta x_2, \Delta y_2)^T \tag{25}$$

$$V_2 = (\Delta u_3, \Delta v_3, \Delta x_3, \Delta y_3)^T \tag{26}$$

where $\Delta u_2 = (u_2 - u_1)$, $\Delta u_3 = (u_3 - u_1)$ and similarly for v, x, and y. We will assume that these vectors define a tangent plane at the center of gravity of the triangle. This consideration has no significance in the Cauchy-Riemann equations but will be important when differential equations with nonconstant coefficients are considered. Contracting[3] these vectors with the differential forms, we get

$$\phi(V_1, V_2) = \frac{1}{2}\tilde{d}u \wedge \tilde{d}y(V_1, V_2) - \frac{1}{2}\tilde{d}v \wedge \tilde{d}x(V_1, V_2) \tag{27}$$

$$\psi(V_1, V_2) = \frac{1}{2}\tilde{d}v \wedge \tilde{d}y(V_1, V_2) + \frac{1}{2}\tilde{d}u \wedge \tilde{d}x(V_1, V_2) \quad . \tag{28}$$

Using the formula $\alpha_1\,\alpha_2(V_1, V_2) = \alpha_1(V_1)\alpha_2(V_2) - \alpha_1(V_2)\alpha_2(V_1)$ where α_1 and α_2 are one-forms, we obtain, after some rearrangement,

$$\Phi_T = \phi(V_1, V_2) = \frac{1}{2}\sum_{i \in j_T} u_i \Delta y_i - \frac{1}{2}\sum_{i \in j_T} v_i \Delta x_i \tag{29}$$

$$\Psi_T = \psi(V_1, V_2) = \frac{1}{2}\sum_{i \in j_T} v_i \Delta y_i + \frac{1}{2}\sum_{i \in j_T} u_i \Delta x_i \tag{30}$$

[3] Applying (V_2, V_1) to the forms will give a sign change.

where $j_T = \{1,2,3\}$ and $\Delta()_i$ denotes a difference taken counterclockwise along the side opposite to j. These can be obtained also by integrating equations (19) and (20) over a triangular element in (x,y)-plane assuming the linear variations of u and v over the element, i.e. the fluctuations. It is now clear that the fluctuations represent the geometric conditions on the solution surface.

To write them compact, as in [1], we introduce the vector $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\mathbf{v}$, $\mathbf{x}$, and $\mathbf{y}$ similarly, and the tensor of 2nd-rank P_2 defined by

$$P_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad . \tag{31}$$

The fluctuations are then written

$$\Phi_T = P_2\mathbf{u}\mathbf{y} - P_2\mathbf{v}\mathbf{x} \tag{32}$$

$$\Psi_T = P_2\mathbf{v}\mathbf{y} + P_2\mathbf{u}\mathbf{x} \quad . \tag{33}$$

It is immediate to notice that each term represents the area of a triangular element projected onto various two-dimensional space. For example $P_2\mathbf{u}\mathbf{y}$ is the area projected onto (u,y)-space. This is a fundamental property of differential forms.

4 Cauchy-Riemnann equations in 3D

Consider the Cauchy-Riemann system extended to three-dimension

$$u_x + v_y + w_z = 0 \tag{34}$$

$$v_x - u_y = 0 \tag{35}$$

$$w_x - u_z = 0 \tag{36}$$

where u, v and w are the velocity components in an incompressible and irrotational flow. We define the three-forms ϕ, ψ and ω as

$$\phi = \frac{1}{6}\tilde{d}u \wedge \tilde{d}y \wedge \tilde{d}z - \frac{1}{6}\tilde{d}v \wedge \tilde{d}x \wedge \tilde{d}z + \frac{1}{6}\tilde{d}w \wedge \tilde{d}x \wedge \tilde{d}y \tag{37}$$

$$\psi = \frac{1}{6}\tilde{d}v \wedge \tilde{d}y \wedge \tilde{d}z + \frac{1}{6}\tilde{d}u \wedge \tilde{d}x \wedge \tilde{d}z \tag{38}$$

$$\omega = \frac{1}{6}\tilde{d}w \wedge \tilde{d}y \wedge \tilde{d}z - \frac{1}{6}\tilde{d}u \wedge \tilde{d}x \wedge \tilde{d}y \tag{39}$$

which obviously constitute a closed ideal. As in the two-dimensional case, the equivalence to the original equations can be verified by imposing the independence of x, y and z. We substitute $\tilde{d}u = u_x\,\tilde{d}x + u_y\,\tilde{d}y + u_z\,\tilde{d}z$ and similar expressions for $\tilde{d}v$ and $\tilde{d}w$ into the three-forms to get

$$\phi = \frac{1}{6}(u_x + v_y + w_z)\,\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z \tag{40}$$

$$\psi = \frac{1}{6}(v_x - u_y)\,\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z \tag{41}$$

$$\omega = \frac{1}{6}(w_x - u_z)\,\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z \quad . \tag{42}$$

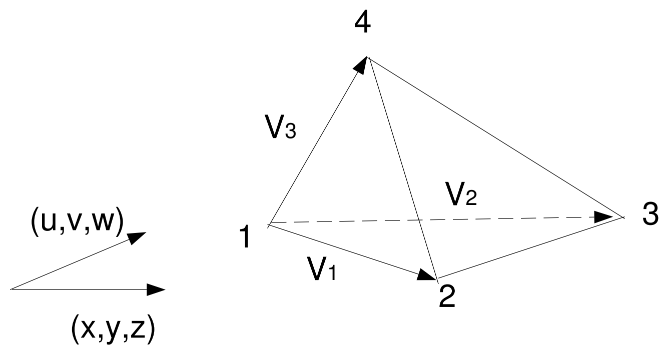

Figure 5: A tetrahedron $T_e \in \{T_e\}$ and tangent vectors in 6-dimensional space.

Manifestly vanishing coefficients implies the original Cauchy-Riemann system.

The solution to equaions (34), (35) and (36) is in this case a three-dimensional submanifold in a six-dimensional manifold whose coordinates are (u, v, w, x, y, z). The tangent vectors of the integral manifold then annul the three-forms everywhere on it. Such a submanifold can be constructed by a three-dimensional triangulation using tetrahedra. This defines a set of tetrahedra, $\{T_e\}$. A tangent space is now three-dimensional and thus we are in need of three linearly independent vectors at a point in each tetrahedron to define a tangent space. Consider a tetrahedron $T_e \in \{T_e\}$ whose vertices are numbered 1,2,3 and 4 (see Figure 5). Any three linearly independent edge vectors can form the desired vector space. Particularly we define the three tangent vectors as the following three edges of the tetrahetron.

$$V_1 = (\Delta u_2, \Delta v_2, \Delta w_2, \Delta x_2, \Delta y_2, \Delta z_2)^T \tag{43}$$

$$V_2 = (\Delta u_3, \Delta v_3, \Delta w_3, \Delta x_3, \Delta y_3, \Delta z_3)^T \tag{44}$$

$$V_3 = (\Delta u_4, \Delta v_4, \Delta w_4, \Delta x_4, \Delta y_4, \Delta z_4)^T \tag{45}$$

where $\Delta u_2 = (u_2 - u_1)$, $\Delta u_3 = (u_3 - u_1)$ and $\Delta u_4 = (u_4 - u_1)$, and similarly for v, w, x, y and z. We will assume that the vector space they define approximates the exact tangent space at the center of gravity of the tetrahedron. Contracting[4] these vectors with the forms, we get

$$\begin{aligned}\phi(V_1, V_2, V_3) &= \frac{1}{6}\tilde{d}u \wedge \tilde{d}y \wedge \tilde{d}z(V_1, V_2, V_3) - \frac{1}{6}\tilde{d}v \wedge \tilde{d}x \wedge \tilde{d}z(V_1, V_2, V_3) \\ &\quad + \frac{1}{6}\tilde{d}w \wedge \tilde{d}x \wedge \tilde{d}y(V_1, V_2, V_3)\end{aligned} \tag{46}$$

$$\psi(V_1, V_2, V_3) = \frac{1}{6}\tilde{d}v \wedge \tilde{d}y \wedge \tilde{d}z(V_1, V_2, V_3) + \frac{1}{6}\tilde{d}u \wedge \tilde{d}x \wedge \tilde{d}z(V_1, V_2, V_3) \tag{47}$$

$$\omega(V_1, V_2, V_3) = \frac{1}{6}\tilde{d}w \wedge \tilde{d}y \wedge \tilde{d}z(V_1, V_2, V_3) - \frac{1}{6}\tilde{d}u \wedge \tilde{d}x \wedge \tilde{d}y(V_1, V_2, V_3) \quad . \tag{48}$$

It is straightforward but tedious to evaluate these by using the definition. Therefore we instead deduce the resulting expressions from a geometrical consideration. As mentioned in the previous section, the evaluation of a two-form on the two tangent vectors has given

[4] Again the order of applying the vectors matters to the sign change in the result.

the area of the triangle 1-2-3 projected onto a two-dimensional space. Similarly in the present case, we will obtain the volume of the tetrahedron 1-2-3-4 projected onto a three-dimensional space for each term in the equations above. We now introduce a third rank tensor P_3, which computes the volume of a tetrahedron in three-dimension, defined by

$$(P_3)_{ijk} = \begin{cases} \frac{1}{6} & \text{if } ijk \text{ is an even permutation of } 1,2,3,4; \\ -\frac{1}{6} & \text{if } ijk \text{ is an odd permutation of } 1,2,3,4; \\ 0 & \text{otherwise.} \end{cases} \tag{49}$$

We define also the vector $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and analogously for others. We thus arrive at the final expressions of the equations,

$$\Phi_{T_e} = \phi(V_1, V_2, V_3) = P_3\mathbf{uyz} - P_3\mathbf{vxz} + P_3\mathbf{wxy} \tag{50}$$

$$\Psi_{T_e} = \psi(V_1, V_2, V_3) = P_3\mathbf{vyz} + P_3\mathbf{uxz} \tag{51}$$

$$\Omega_{T_e} = \omega(V_1, V_2, V_3) = P_3\mathbf{wyz} - P_3\mathbf{uxy} \quad . \tag{52}$$

These quantities Φ_{T_e}, Ψ_{T_e} and Ω_{T_e} are nothing but the fluctuations. Again the conclusion is that the fluctuations represent the geometrical property of the solution manifold, i.e. tangent vectors annul the three-forms.

5 Concluding Remarks

The derivation of fluctuations in terms of exterior differential forms were described. It was shown that fluctuations in fact represent geometrical properties of integral manifolds. Vanishing fluctuations means that tangent vectors defined at the center of gravity of each element in an approximate integral manifold annul a set of differential forms equivalent to a system of differential equations.

We remark that the fluctuations can be obtained also by integrating equivalent differential forms over an element of a triangulated manifold. This means that the fluctuations can be thought of also as the integral of errors over each element in a triangulation. It is however expected that the use of the integration of differential forms over manifolds would lead us to the usual derivation of fluctuations because such an integration is done by choosing parameters that describe a solution manifold, and transform the integration into the parameter subspace. Indeed if a physical space is chosen for the integration space, it will lead us to a familier derivation of the fluctuations of the original differential equations. Nevertheless it is interesting to note that if dependent variables are chosen instead as parameters, it turns out that the fluctuations are being derived from the hodograph equations. And whatever are chosen as parameters, we will obtain the same fluctuations. Fluctuations being derived in this way, their error estimates can be obtained in a usual manner. It then implies that expressions for errors will be different, depending on parameters chosen for the integration.

Finally, although this study has verified that fluctuations possess a geometrical property, developing a successful numerical algorithm that exploits the fact is another important issue.

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