On Jacobi Iteration for Singular System

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Abstract

1 Introduction

Consider a linear system:

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{x} = [x_1, x_2, x_3, \cdots, x_n]^t$ and $\mathbf{b} = [b_1, b_2, b_3, \cdots, b_n]^t$ are vectors of n components. The system has has a unique solution only if the matrix $\mathbf{A}$ is non-singular. If $\mathbf{A}$ is singular, there are two possibilities: no solutions or infinitely many solutions. Our interest here is to see if the singular system can be solved by the Jacobi iteration in the latter case even though the solution will not be unique and depend on an initial solution.

2 Jacobi Iteration for a Singular Matrix

Consider the Jacobi iteration:

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \omega \mathbf{D}^{-1} \left[\mathbf{b} - \mathbf{A}\mathbf{x}^k \right], \tag{2}$$

where $\mathbf{D}$ denotes the diagonal of $\mathbf{A}$, which is assumed to be invertible, $\mathbf{I}$ is the $n \times n$ identity matrix, $\omega (\leq 1)$ is a positive under-relaxation parameter, and k is the iteration counter. Let us write the iteration as

$$\mathbf{x}^{k+1} = \mathbf{T}\mathbf{x}^k + \omega \mathbf{D}^{-1}\mathbf{b}, \tag{3}$$

where $\mathbf{T}$ is the iteration matrix:

$$\mathbf{T} = \mathbf{I} - \omega \mathbf{D}^{-1}\mathbf{A}. \tag{4}$$

The iteration converges if the spectral radius of $\mathbf{T}$ is less than one. However, for a singular matrix $\mathbf{A}$, it will be one. It can easily shown as follows. If $\mathbf{A}$ is singular, there is at least one zero eigenvalue. Suppose the i-th eigenvalue is zero:

$$\mathbf{A}\mathbf{r}_i = 0, \tag{5}$$

where $\mathbf{r}_i$ is the i-th eigenvector. Then, we have

$$-\omega \mathbf{D}^{-1}\mathbf{A}\mathbf{r}_i = 0, \tag{6}$$

$$\mathbf{I}\mathbf{r}_i - \omega \mathbf{D}^{-1}\mathbf{A}\mathbf{r}_i = \mathbf{I}\mathbf{r}_i, \tag{7}$$

$$\mathbf{T}\mathbf{r}_i = \mathbf{r}_i, \tag{8}$$

which shows that $\mathbf{T}$ has the eigenvalue of one. However, the iteration converges in a special circumstance.

Let $\mathbf{x}^0$ be the initial solution. Then,

$$\mathbf{x}^1 = \mathbf{T}\mathbf{x}^0 + \omega\mathbf{D}^{-1}\mathbf{b}, \tag{9}$$

$$\begin{aligned} \mathbf{x}^2 &= \mathbf{T}\mathbf{x}^1 + \omega\mathbf{D}^{-1}\mathbf{b} && (10)\\ &= \mathbf{T}(\mathbf{T}\mathbf{x}^0 + \omega\mathbf{D}^{-1}\mathbf{b}) + \omega\mathbf{D}^{-1}\mathbf{b} && (11)\\ &= \mathbf{T}^2\mathbf{x}^0 + \omega\mathbf{T}\mathbf{D}^{-1}\mathbf{b} + \omega\mathbf{D}^{-1}\mathbf{b}. && (12) \end{aligned}$$

$$\begin{aligned} \mathbf{x}^3 &= \mathbf{T}\mathbf{x}^2 + \omega\mathbf{D}^{-1}\mathbf{b} && (13)\\ &= \mathbf{T}(\mathbf{T}^2\mathbf{x}^0 + \omega\mathbf{T}\mathbf{D}^{-1}\mathbf{b} + \omega\mathbf{D}^{-1}\mathbf{b}) + \omega\mathbf{D}^{-1}\mathbf{b} && (14)\\ &= \mathbf{T}^3\mathbf{x}^0 + \omega\mathbf{T}^2\mathbf{D}^{-1}\mathbf{b} + \omega\mathbf{T}\mathbf{D}^{-1}\mathbf{b} + \omega\mathbf{D}^{-1}\mathbf{b}. && (15) \end{aligned}$$

and thus

$$\mathbf{x}^k = \mathbf{T}^k\mathbf{x}^0 + \omega\left[\sum_{m=0}^{k-1}\mathbf{T}^m\right]\mathbf{D}^{-1}\mathbf{b}. \tag{16}$$

Assume that $\mathbf{T}$ has a complete set of linearly independent eigenvectors:

$$\mathbf{T} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{L}, \tag{17}$$

where $\mathbf{R}$ is the right eigenvector matrix, $\mathbf{L} = \mathbf{R}^{-1}$, and $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues. Substitute this into Equation (16) to get

$$\begin{aligned} \mathbf{x}^k &= \mathbf{R}\boldsymbol{\Lambda}^k\mathbf{L}\mathbf{x}^0 + \omega\sum_{m=0}^{k-1}\mathbf{R}\boldsymbol{\Lambda}^m\mathbf{L}\mathbf{D}^{-1}\mathbf{b} && (18)\\ &= \sum_{i=1}^{n}\lambda_i^k(\boldsymbol{\ell}_i\mathbf{x}^0)\mathbf{r}_i + \omega\left[\sum_{m=0}^{k-1}\left\{\sum_{i=1}^{n}\lambda_i^m(\boldsymbol{\ell}_i\mathbf{D}^{-1}\mathbf{b})\mathbf{r}_i\right\}\right], && (19) \end{aligned}$$

where λ_i is the i-th eigenvalue, $\boldsymbol{\ell}_i$ is the i-th left eigenvector, and $\mathbf{r}_i$ is the i-th right eigenvector. In the first sum, all terms with $|\lambda_i^k| < 1$ will vanish for $n \to \infty$. If we assume there are no eigenvalues larger than 1, then, the remaining eigenvalues have the value of 1.

$$\mathbf{x}^k = \sum_{i\in\{i:\lambda_i=1\}}(\boldsymbol{\ell}_i\mathbf{x}^0)\mathbf{r}_i + \omega\left[\sum_{m=0}^{k-1}\left\{\sum_{i=1}^{n}\lambda_i^m(\boldsymbol{\ell}_i\mathbf{D}^{-1}\mathbf{b})\mathbf{r}_i\right\}\right]. \tag{20}$$

The first term is constant. The second term contains terms that converge (i.e., $|\lambda_i| < 1$) and those that diverge, i.e., $\lambda_i = 1$. But it will converge if $\boldsymbol{\ell}_i\mathbf{D}^{-1}\mathbf{b} = 0$ for all i with $\lambda_i = 1$:

$$\mathbf{x}^k = \sum_{i\in\{i:\lambda_i=1\}}(\boldsymbol{\ell}_i\mathbf{x}^0)\mathbf{r}_i + \omega\left[\sum_{m=0}^{k-1}\left\{\sum_{i\in\{i:|\lambda_i|<1\}}\lambda_i^m(\boldsymbol{\ell}_i\mathbf{D}^{-1}\mathbf{b})\mathbf{r}_i\right\}\right]. \tag{21}$$

This shows that the solution will depend on the initial solution.

To see if it satisfies $\mathbf{A}\mathbf{x} = \mathbf{b}$, consider

$$\mathbf{A}\mathbf{x}^k = \sum_{i\in\{i:\lambda_i=1\}}(\boldsymbol{\ell}_i\mathbf{x}^0)\mathbf{A}\mathbf{r}_i + \omega\left[\sum_{m=0}^{k-1}\left\{\sum_{i\in\{i:|\lambda_i|<1\}}\lambda_i^m(\boldsymbol{\ell}_i\mathbf{D}^{-1}\mathbf{b})\mathbf{A}\mathbf{r}_i\right\}\right]. \tag{22}$$

The first term vanishes because $\mathbf{r}_i$ for $\lambda_i = 1$ corresponds to the eigenvector of $\mathbf{A}$ with zero eigenvalue as shown earlier:

$$\mathbf{A}\mathbf{x}^k = \omega\left[\sum_{m=0}^{k-1}\left\{\sum_{i\in\{i:|\lambda_i|<1\}}\lambda_i^m(\boldsymbol{\ell}_i\mathbf{D}^{-1}\mathbf{b})\mathbf{A}\mathbf{r}_i\right\}\right]. \tag{23}$$

To eliminate $\mathbf{A}$, consider

$$\mathbf{T}\mathbf{r}_i = \lambda_i \mathbf{r}_i, \tag{24}$$

$$(\mathbf{I} - \omega \mathbf{D}^{-1}\mathbf{A})\mathbf{r}_i = \lambda_i \mathbf{r}_i, \tag{25}$$

$$\omega \mathbf{D}^{-1}\mathbf{A}\mathbf{r}_i = (1 - \lambda_i)\mathbf{r}_i, \tag{26}$$

$$\mathbf{A}\mathbf{r}_i = \frac{1 - \lambda_i}{\omega}\mathbf{D}\mathbf{r}_i, \tag{27}$$

and substitute this into Equation (22) to get

$$\begin{aligned}
\mathbf{A}\mathbf{x}^k &= \sum_{m=0}^{k-1} \left\{ \sum_{i \in \{i:|\lambda_i|<1\}} (1-\lambda_i)\lambda_i^m (\boldsymbol{\ell}_i \mathbf{D}^{-1}\mathbf{b})\mathbf{D}\mathbf{r}_i \right\} && (28)\\
&= \sum_{i \in \{i:|\lambda_i|<1\}} \left\{ \sum_{m=0}^{k-1} (1-\lambda_i)\lambda_i^m (\boldsymbol{\ell}_i \mathbf{D}^{-1}\mathbf{b})\mathbf{D}\mathbf{r}_i \right\} && (29)\\
&= \sum_{i \in \{i:|\lambda_i|<1\}} \left\{ \frac{1 + \lambda_i + \lambda_i^2 + \lambda_i^3 + \cdots + \lambda_i^{k-1}}{1 + \lambda_i + \lambda_i^2 + \lambda_i^3 + \cdots} (\boldsymbol{\ell}_i \mathbf{D}^{-1}\mathbf{b})\mathbf{D}\mathbf{r}_i \right\} && (30)\\
&\to \sum_{i \in \{i:|\lambda_i|<1\}} \left\{ (\boldsymbol{\ell}_i \mathbf{D}^{-1}\mathbf{b})\mathbf{D}\mathbf{r}_i \right\} \quad \text{as} \quad k \to \infty, && (31)
\end{aligned}$$

which can be $\mathbf{b}$ and thus $\mathbf{A}\mathbf{x}^k = \mathbf{b}$ if

$$\mathbf{b} = \sum_{i \in \{i:|\lambda_i|<1\}} (\boldsymbol{\ell}_i \mathbf{D}^{-1}\mathbf{b})\mathbf{D}\mathbf{r}_i. \tag{32}$$

Note that $\boldsymbol{\ell}_i \mathbf{D}^{-1}$ and $\mathbf{D}\mathbf{r}_i$ are the left and right eigenvectors of $\mathbf{D}\mathbf{A}\mathbf{D}^{-1} (\neq \mathbf{A})$.

If $\mathbf{D} = d\mathbf{I}$, i.e., all diagonal elements are the same, then Equation (32) means that $\mathbf{b}$ exists in the space of the right eigenvectors of $|\lambda_i| < 1$:

$$\mathbf{b} = \sum_{i \in \{i:|\lambda_i|<1\}} (\boldsymbol{\ell}_i \mathbf{b})\mathbf{r}_i. \tag{33}$$

Furthermore, in this case, $\mathbf{r}_i$ and $\boldsymbol{\ell}_i$ are the left and right eigenvectors of $\mathbf{A}$, corresponding to its nonzero eigenvalues.

3 Example

A simple example is the following:

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \tag{34}$$

Clearly, $\mathbf{A}$ is singular and the system admits infinitely many solutions. The matrix $\mathbf{A}$ has the following eigen-decomposition:

$$\mathbf{A} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{L}, \tag{35}$$

where

$$\mathbf{R} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}. \tag{36}$$

The iteration matrix is given by

$$\mathbf{T} = \mathbf{I} - \omega \mathbf{D}^{-1}\mathbf{A} = \begin{bmatrix} 1-\omega & -\omega \\ -\omega & 1-\omega \end{bmatrix}, \tag{37}$$

which gives

$$\mathbf{T} = \mathbf{R}\mathbf{\Lambda}\mathbf{L}, \tag{38}$$

where

$$\mathbf{R} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1-2\omega \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}. \tag{39}$$

For the Jacobi iteration to converge, we need to choose ω such that

$$0 < \omega < 1. \tag{40}$$

And since $\mathbf{b}$ is in the space of the right eigenvectors, the Jacobi iteration converges to

$$\mathbf{x}^k \quad = \quad \boldsymbol{\ell}_1 \mathbf{x}^0 \mathbf{r}_1 + \omega \left[\sum_{m=0}^{k-1} \left\{ \lambda_2^m (\boldsymbol{\ell}_2 \mathbf{D}^{-1}\mathbf{b}) \mathbf{r}_2 \right\} \right]. \tag{41}$$

For example, for $\mathbf{x}^0 = (0,0)$ and $k = 25$, this gives

$$\mathbf{x}^k = (0.499999147090913, 0.499999147090913), \tag{42}$$

and for $\mathbf{x}^0 = (1,5)$ and $k = 25$,

$$\mathbf{x}^k = (-1.50000085290909, 2.49999914709091), \tag{43}$$

both of which are solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$.