L_2 Error Estimate for Triangular Grids

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1 L_2 Error for Quadratic Functions

Suppose we have a quadratic function $f(x,y)$ in a domain Ω and its piecewise linear continuous approximation of unknown accuracy, i.e. triangulation of the quadratic function. We are going to compute its local L_2 error on a triangle exactly. Let $\{T\}$ denote the set of triangles that fills Ω, and $T \in \{T\}$ be formed by vertices 1,2 and 3. We define the local L_2 error on triangle T by

$$\left(L_2^T\right)^2 = \int_T \left[f(x,y) - u^T(x,y)\right]^2 dx\, dy \ . \tag{1}$$

Here $u^T(x,y)$ is the restrition of the approximation to triangle T which can be written as

$$u^T(x,y) = \overline{u^T} + \frac{\partial u^T}{\partial x}\Delta x_c + \frac{\partial u^T}{\partial y}\Delta y_c \tag{2}$$

where

$$\frac{\partial u^T}{\partial x} = \frac{1}{2S^T}\sum_{i=1}^{3} u_i \Delta y_i, \qquad \frac{\partial u^T}{\partial y} = -\frac{1}{2S^T}\sum_{i=1}^{3} u_i \Delta x_i, \tag{3}$$

u_i are approximations of f at vertices, $\overline{u^T} = (u_1+u_2+u_3)/3$, S^T is the area of T, $x_c = (x_1+x_2+x_3)/3$, $\Delta x_c = x - x_c$, Δx_i is the difference of x taken counterclockwise along the side opposite to j, and similarly for y. Being a quadratic function, $f(x,y)$ can also be written similarly in the form of Taylor series around the centroid of the triangle.

$$f(x,y) = f(x_c,y_c) + p_c\Delta x_c + q_c\Delta y_c + \frac{1}{2}\left[\Delta x_c\frac{\partial}{\partial x} + \Delta y_c\frac{\partial}{\partial y}\right]^2 f \tag{4}$$

where

$$p_c = \frac{\partial}{\partial x} f(x_c,y_c), \qquad q_c = \frac{\partial}{\partial y} f(x_c,y_c). \tag{5}$$

The difference then becomes

$$f(x,y) - u^T(x,y) = f(x_c,y_c) - \overline{u^T} + \Psi_x^T\Delta x_c + \Psi_y^T\Delta y_c + \frac{1}{2}\left[\Delta x_c\frac{\partial}{\partial x} + \Delta y_c\frac{\partial}{\partial y}\right]^2 f \tag{6}$$

where

$$\Psi_x^T = p_c - \frac{\partial u^T}{\partial x}, \qquad \Psi_y^T = q_c - \frac{\partial u^T}{\partial y}. \tag{7}$$

The first two terms can be written as

$$f(x_c,y_c) - \overline{u^T} = \overline{e^T} - \frac{1}{3\cdot 2}\left[\Delta x_{ic}\frac{\partial}{\partial x} + \Delta y_{ic}\frac{\partial}{\partial y}\right]^2 f \tag{8}$$

where $\overline{e^T}$ is the average of the nodal errors defined by $\frac{1}{3}\sum_{i=1}^{3}\{f(x_i,y_i)-u_i\}$, and $\Delta x_{ic}=x_i-x_c$, $\Delta y_{ic}=y_i-y_c$. Therefore, introducing the notation,

$$\begin{aligned} S_c(\Delta x_c,\Delta y_c) &= \Psi_x^T\Delta x_c+\Psi_y^T\Delta y_c \\ Q_c(\Delta x_c,\Delta y_c) &= \frac{1}{2}\left[\Delta x_c\frac{\partial}{\partial x}+\Delta y_c\frac{\partial}{\partial y}\right]^2 f \\ Q_{ic} &= \frac{1}{6}\sum_{i=1}^{3}\left[\Delta x_{ic}\frac{\partial}{\partial x}+\Delta y_{ic}\frac{\partial}{\partial y}\right]^2 f \end{aligned}$$

we have

$$f(x,y)-u^T(x,y)=\overline{e^T}+S_c+Q_c-Q_{ic}. \tag{9}$$

Now the local L_2 error can be written as

$$\left(L_2^T\right)^2=\int_T\left[\overline{e^T}^2+S_c^2+Q_c^2+Q_{ic}^2+2S_cQ_c-2Q_cQ_{ic}+2\overline{e^T}\left(Q_c-Q_{ic}\right)-2S_cQ_{ic}+2\overline{e^T}S_c\right]dx\,dy \tag{10}$$

The integrals of the last two terms identically vanish, and we are left with

$$\begin{aligned} \left(L_2^T\right)^2 = & \int_T\overline{e^T}^2\,dx\,dy+2\int_T\overline{e^T}\left(Q_c-Q_{ic}\right)\,dx\,dy+\int_T S_c^2\,dx\,dy+\int_T Q_c^2\,dx\,dy \\ + & \int_T Q_{ic}^2\,dx\,dy+2\int_T rS_cQ_c\,dx\,dy-2\int_T Q_cQ_{ic}\,dx\,dy. \end{aligned} \tag{11}$$

Integrating, we obtain

$$\begin{aligned} \left(L_2^T\right)^2 = & \overline{e^T}^2S^T-\overline{e^T}\frac{S^T}{9}\sum_{i=1}^{3}\left(\Delta x_i\frac{\partial}{\partial x}+\Delta y_i\frac{\partial}{\partial y}\right)^2 f-\frac{1}{135}{S^T}^2S_{pq}^T \\ + & \frac{S^T}{36}\sum_{i=1}^{3}\left(\Psi_x^T\Delta x_i+\Psi_y^T\Delta y_i\right)^2+\frac{S^T}{2160}\left\{\sum_{i=1}^{3}\left(\Delta x_i\frac{\partial}{\partial x}+\Delta y_i\frac{\partial}{\partial y}\right)^2 f\right\}^2 \\ + & \frac{S^T}{324}\left\{\sum_{i=1}^{3}\left(\Delta x_i\frac{\partial}{\partial x}+\Delta y_i\frac{\partial}{\partial y}\right)^2 f\right\}^2 \\ + & \frac{S^T}{30}\sum_{i=1}^{3}\left(\Psi_x^T\Delta x_{ic}+\Psi_y^T\Delta y_{ic}\right)\left(\Delta x_{kc}\frac{\partial}{\partial x}+\Delta y_{kc}\frac{\partial}{\partial y}\right)\left(\Delta x_{lc}\frac{\partial}{\partial x}+\Delta y_{lc}\frac{\partial}{\partial y}\right)f \\ - & \frac{S^T}{648}\left\{\sum_{i=1}^{3}\left(\Delta x_i\frac{\partial}{\partial x}+\Delta y_i\frac{\partial}{\partial y}\right)^2 f\right\}^2 \end{aligned}$$

where

$$S_{pq}^T=\frac{1}{2}\sum_{i=1}^{3}p_i\Delta q_i, \tag{12}$$

and it simplifies to

$$\begin{aligned} \left(L_2^T\right)^2 = & \overline{e^T}^2S^T-\overline{e^T}\frac{S^T}{9}\sum_{i=1}^{3}\left(\Delta x_i\frac{\partial}{\partial x}+\Delta y_i\frac{\partial}{\partial y}\right)^2 f-\frac{1}{135}{S^T}^2S_{pq}^T \\ + & \frac{S^T}{36}\sum_{i=1}^{3}\left(\Psi_x^T\Delta x_i+\Psi_y^T\Delta y_i\right)^2+\frac{13}{6480}\left\{\sum_{i=1}^{3}\left(\Delta x_i\frac{\partial}{\partial x}+\Delta y_i\frac{\partial}{\partial y}\right)^2 f\right\}^2 S^T \\ + & \frac{S^T}{30}\sum_{i=1}^{3}\left(\Psi_x^T\Delta x_{ic}+\Psi_y^T\Delta y_{ic}\right)\left(\Delta x_{kc}\frac{\partial}{\partial x}+\Delta y_{kc}\frac{\partial}{\partial y}\right)\left(\Delta x_{lc}\frac{\partial}{\partial x}+\Delta y_{lc}\frac{\partial}{\partial y}\right)f \end{aligned}$$

where the subscripts k and l take 1,2,3, and are permuted cyclically for i. To simplify this further, we rewrite the curvature term as follows.

$$\begin{aligned}\sum_{i=1}^{3}\left(\Delta x_i\frac{\partial}{\partial x}+\Delta y_i\frac{\partial}{\partial y}\right)^2 f &= \sum_{i=1}^{3}\Delta x_i^2\frac{\partial^2 f}{\partial x^2}+2\sum_{i=1}^{3}\Delta x_i\Delta y_i\frac{\partial^2 f}{\partial x\partial y}+\sum_{i=1}^{3}\Delta y_i^2\frac{\partial^2 f}{\partial y^2}\\ &= \sum_{i=1}^{3}\Delta x_i^2\frac{\partial p}{\partial x}+\sum_{i=1}^{3}\Delta x_i\Delta y_i\frac{\partial p}{\partial y}+\sum_{i=1}^{3}\Delta x_i\Delta y_i\frac{\partial q}{\partial x}+\sum_{i=1}^{3}\Delta y_i^2\frac{\partial q}{\partial y}\\ &= \sum_{i=1}^{3}\Delta x_i\left(\frac{\partial p}{\partial x}\Delta x_i+\frac{\partial p}{\partial y}\Delta y_i\right)+\sum_{i=1}^{3}\Delta y_i\left(\frac{\partial q}{\partial x}\Delta x_i+\frac{\partial q}{\partial y}\Delta y_i\right).\end{aligned}$$

Now, for quadratic functions the following is an identity.

$$\Delta p_i=\frac{\partial p}{\partial x}\Delta x_i+\frac{\partial p}{\partial y}\Delta y_i \tag{13}$$

and similarly for q. Hence we have

$$\sum_{i=1}^{3}\left(\Delta x_i\frac{\partial}{\partial x}+\Delta y_i\frac{\partial}{\partial y}\right)^2 f=\sum_{i=1}^{3}\left(\Delta p_i\Delta x_i+\Delta q_i\Delta y_i\right). \tag{14}$$

Using this exact relation, we obtain

$$\begin{aligned}\frac{1}{S^T}\left(L_2^T\right)^2 = \quad & \overline{e^T}^2-\frac{\overline{e^T}}{9}\sum_{i=1}^{3}\left(\Delta p_i\Delta x_i+\Delta q_i\Delta y_i\right)+\frac{1}{36}\sum_{i=1}^{3}\left(\Psi_x^T\Delta x_i+\Psi_y^T\Delta y_i\right)^2\\ + \quad & \frac{1}{90}\sum_{i=1}^{3}\left(\Psi_x^T\Delta x_{ic}+\Psi_y^T\Delta y_{ic}\right)\left\{\Delta x_{kc}\left(\Delta p_k-\Delta p_i\right)+\Delta y_{kc}\left(\Delta q_k-\Delta q_i\right)\right\}\\ + \quad & \frac{13}{6480}\left\{\sum_{i=1}^{3}\left(\Delta p_i\Delta x_i+\Delta q_i\Delta y_i\right)\right\}^2-\frac{1}{135}S^T S_{pq}^T.\end{aligned} \tag{15}$$

We first consider the interpolation error, i.e. exact nodal values. In this case, it can be shown that this simplifies to

$$\left(L_2^T\right)^2=\frac{C_T^2 S^T}{2160} \tag{16}$$

where

$$C_T=\sqrt{\sum_{i=1}^{3}K_i^2+4\left\{\sum_{i=1}^{3}K_i\right\}^2-16\,S^T S_{pq}^T}, \tag{17}$$

and

$$K_i=\Delta p_i\Delta x_i+\Delta q_i\Delta y_i. \tag{18}$$

Adding this over the set of triangles $\{T\}$ and taking square root, we obtain the following L_2 interpolation error

$$L_2(\Omega)=\sqrt{\sum_{T\in\{T\}}\frac{C_T^2 S^T}{2160}}. \tag{19}$$

This is exact for quadratic functions and therefore second-order accurate for nonquadratic functions.

Next let us consider the case that the nodal values are not exact. In particular, we assume that the nodal values are numerical solutions of some partial differential equation. To begin with, we assume also that the exact solution is quadratic, so that (15) remains exact. In such cases, it can be deduced from (15)

N_T	L	L_2	(18)	Relative Error
1	1.00000000E-01	1.40670589E-02	1.21741840E-02	1.34560815E-01
4	5.00000000E-02	3.60805175E-03	3.48917937E-03	3.29464182E-02
16	2.50000000E-02	9.07202593E-04	8.99727380E-04	8.23985002E-03
64	1.25000000E-02	2.27187212E-04	2.26718995E-04	2.06092881E-03
256	6.25000000E-03	5.68218214E-05	5.67925412E-05	5.15299048E-04

Table 1: Equilateral triangles

N_T	L	L_2	(19)	Relative Error
1	7.01662521E-02	1.78765737E-03	1.53570624E-03	1.40939275E-01
4	3.50831260E-02	7.17631991E-04	6.95187049E-04	3.12763950E-02
16	1.75415630E-02	1.85849668E-04	1.84414102E-04	7.72434133E-03
64	8.77078151E-03	4.68297638E-05	4.67395685E-05	1.92602517E-03
256	4.38539075E-03	1.17300239E-05	1.17243794E-05	4.81205535E-04

Table 2: Skew triangles

that the interpolation error (18) estimates the L_2 error with first-order accuracy if the nodal values are third-order, which is valid also for nonquadratic solutions. This implies that the interpolation error would not be a useful error estimator in practical applications because the second-order schemes are prevalant. However the interpolation error is still an important quantity. It represents the error components that are irrelevant to numerical schemes. In other words, it can be controlled only by the grid configuration.

2 Numerical Tests

In this section, some numerical results are shown that verify the analysis in the previous section. We use the following exponential function for this test.

$$f(x,y) = \exp\{-100(x^2+y^2)\} \tag{20}$$

The first derivatives are computed by a second-order difference approximation with $h = 1.0E-06$. The domain is a triangle and is systematically refined by connecting the mid points of the sides. The results shown in Table 1 and 2 verify the second order accuracy of the interpolation error (18). In the tables, N_T is the number of the triangles, L is the arithmetic average of the sides of the triangle, and L_2 is the actual L_2 error computed by the seven-point Gaussian quadrature. Table 1 shows the result for the domain of the equilateral triangle with the centroid placed at the origin and the side length 0.1. Table 2 shows the result for the domain of a skew triangle defined by the vertices $(0.0, 0.0)$, $(0.01, 0.0)$ and $(0.0, 0.1)$. In both cases, the second-order convergence can be observed. See also Figures (1) and (2).

3 Grid Adaptation

Now, we consider the use of the error estimate for solving partial differential equations. Here we do not consider refining a grid but consider improving an existing grid. A simple strategy is to move the nodes such that the local area-weighted average error, i.e. C_T, is equidistributed over the elements. This can be done by minimizing

$$\mathcal{F} = \sum_{T\in\{T\}} F_T = \frac{1}{2}\sum_{T\in\{T\}} \frac{\Omega_T^2}{S^T} \tag{21}$$

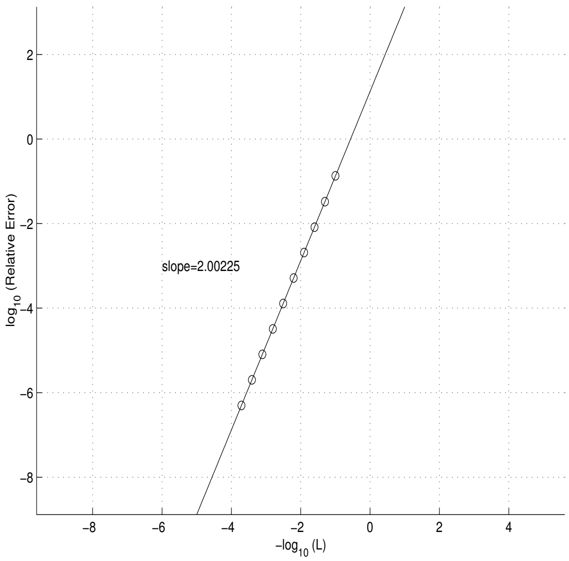

Figure 1: $-log_{10}(L)$ vs. log_{10} (Relative Error) and the linear least-squares fits. Equilateral Triangles.

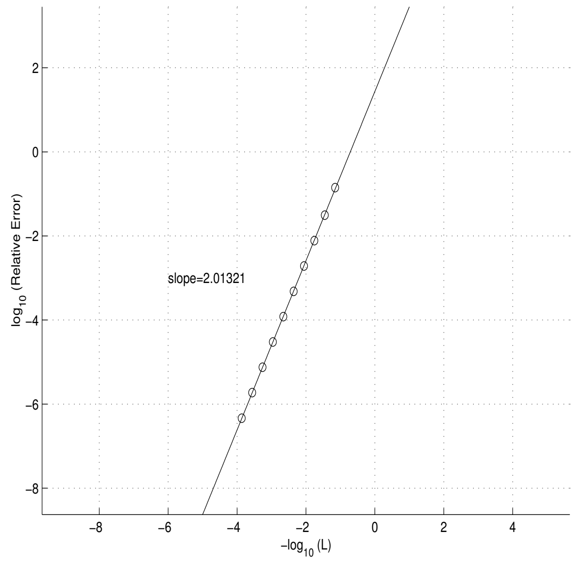

Figure 2: $-log_{10}(L)$ vs. log_{10} (Relative Error) and the linear least-squares fits. Skew Triangles.

with respect to the nodal coordinates where $\Omega = C_T - C$ and C is the arithmetic average of C_T. Note that we are actually trying to equidistribute C_T^2. Using the steepest descent method, we have the adjustment to the node j,

$$\delta \mathbf{x}_j = -\omega \frac{\partial \mathcal{F}}{\partial \mathbf{x}_j} = -\omega \sum_{T \in \{T_j\}} \frac{1}{S^T} \left[\frac{\partial \Omega_T}{\partial \mathbf{x}_j} \Omega_T + \frac{1}{2} \mathbf{n}_T F_T \right] \tag{22}$$

where $\mathbf{n}_T$ is the outward normal vector of the edge opposite to the node j with its magnitude equal to the length of the edge. The derivative of Ω_T is given by

$$\frac{\partial \Omega_T}{\partial \mathbf{x}_j} = 2\left(4K_T + K_l\right) \left[\begin{array}{c} \Delta p_l \\ \Delta q_l \end{array} \right] - 2\left(4K_T + K_r\right) \left[\begin{array}{c} \Delta p_r \\ \Delta q_r \end{array} \right] + 8 S_{pq} \mathbf{n}_T \tag{23}$$

where

$$K_T = \sum_{i=1}^{3} K_i. \tag{24}$$

To perform this minimization, we need the first derivatives, p and q at nodes. These may be obtained by some averages of the derivatives of the numerical solution. For example, we can use area-weighted averages of the derivatives,

$$p_j = \frac{\sum_{T \in \{T_j\}} \frac{\partial u^T}{\partial x} S^T}{\sum_{T \in \{T_j\}} S^T}, \quad q_j = \frac{\sum_{T \in \{T_j\}} \frac{\partial u^T}{\partial y} S^T}{\sum_{T \in \{T_j\}} S^T}. \tag{25}$$