On the Equivalence of LSM and FEM

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July, 1998

Abstract

In this paper, we show that least-squares method for the Cauchy-Riemann system and finite element method for the associated Laplace equations are equivalent.

1 Cauchy-Riemann System

We seek u and v that satisfy

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{2}$$

in a domain Ω, with the boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s} \tag{3}$$

$$v = g(x,y) \tag{4}$$

on the boundary $\partial\Omega$, where $g(x,y)$ is a given function, $\frac{\partial}{\partial n}$ is the derivative normal to the boundary and $\frac{\partial}{\partial s}$ is the derivative tangent to the boundary. It is well-known that the Cauchy-Riemann system is equivalent to the two Laplace equations defined by

$$-\nabla^2 u = 0 \tag{5}$$

$$-\nabla^2 v = 0 \tag{6}$$

in Ω with the same boundary conditions. Therefore the solutions must be the same irrespective of the choice of the governing equations. In this paper, we will show that least-squares method applied to the Cauchy-Riemann equations is equivalent to finite element method applied to the two Laplace equations, i.e. they give the identical system to solve and thus the same solutions.

2 Basis Functions

Although it is not necessary to introduce the concept of basis functions in the case of least-squares method, it may be useful to define local piecewise linear interpolating functions when the two methods are compared. For this purpose, here we will establish the relation between the two functions.

We begin by triangulating the domain Ω into a set of triangles $\{T\}$ and nodes $\{J\}$ consisting in the interior nodes $\{J_i\}$ and the boundary nodes $\{J_b\}$. A group of triangles that shares node j is denoted by $\{T_j\}$, equivalently this local domain will be denoted also by Ω_j with its boundary being $\partial\Omega_j$. We also introduce the notations N_i and N_b for the number of interior nodes and boundary nodes respectively. The total number of nodes N is then equal to $N_i + N_b$.

In finite element method, the solution is sought in a vector subspace which is spanned by a set of basis functions denoted by $\phi_j(x,y)$ $j = 1, 2, ...N$. Each $\phi_j(x,y)$ is unity at the vertex j , zero on $\partial\Omega_j$, and varies linearly within each element belonging to $\{T_j\}$. The approximate solutions u_h and v_h are then represented by

$$u_h = \sum_{i\in\{J\}} u_i\,\phi_i(x,y) \tag{7}$$

$$v_h = \sum_{i\in\{J\}} v_i\,\phi_i(x,y) \tag{8}$$

where u_i and v_i are the nodal values of u and v. On the other hand, the functions useful in least-squares method are piecewise linear interpolating functions $\psi_i^T(x,y)$ defined within each element where $\psi_i^T(x,y)$ is a linear function that takes the value unity at the vertex i and zero at the others. The solutions are then written within each element

$$u^T = \sum_{i\in\{i_T\}} u_i\,\psi_i^T(x,y) \tag{9}$$

$$v^T = \sum_{i\in\{i_T\}} v_i\,\psi_i^T(x,y). \tag{10}$$

where $\{i_T\}$ is a set of vertices that defines the element T. Important relations are

$$u^T = u_h|_T, \quad v^T = v_h|_T \tag{11}$$

where $|_T$ means the restriction on the element T.

3 Finite Element Method

In this section, we will give variational formulations for the pair of Laplace equations (5) and (6), and the corrseponding finite element discretizations using piecewise linear elements.

The Variational Formulation:

Let U and V be the trial spaces for u and v respectively defined by

$$U = \{u \in H^1(\Omega)\} \tag{12}$$

$$V = \{v \in H^1(\Omega) \mid v = g(x,y) \text{ on } \partial\Omega\} \tag{13}$$

where U will be used also as the test space for u. For v, we define the test space V_0 by

$$V_0 = \{v_0 \in H^1(\Omega) \mid v_0 = 0 \text{ on } \partial\Omega\}. \tag{14}$$

The variational forms are easily found by multiplying the Laplace equations by test functions $u_0 \in U$ and $v_0 \in V_0$, integrating over Ω, and by using Green's formula. We are then led to the following variational formulations: find $u \in U$ and $v \in V$ such that

$$\iint_\Omega \nabla u\cdot\nabla u_0\; dxdy - \oint_{\partial\Omega} \frac{\partial v}{\partial s} u_0\; ds = 0 \quad \forall u_0 \in U \tag{15}$$

$$\iint_\Omega \nabla v\cdot\nabla v_0\; dxdy = 0 \quad \forall v_0 \in V_0 \tag{16}$$

where we have used the Neuman condition (3) to rewrite the boundary integral for u.

The finite element discretization:

For the finite element space construction we use piecewise linear functions. The finite element spaces for (12), (13) and (14) are denoted by U_h, V_h and V_{0h} respectively, with the basis function associated with node $j \in \{J\}$ denoted by ϕ_j. Choosing the basis function as a test function, we obtain the following discrete problems: find $u_h \in U_h$ and $v_h \in V_h$ such that

$$\iint_\Omega \nabla u_h \cdot \nabla \phi_j \, dxdy - \oint_{\partial\Omega} \frac{\partial v_h}{\partial s} \phi_j \, ds = 0 \quad \forall j \in \{J\} \tag{17}$$

$$\iint_\Omega \nabla v_h \cdot \nabla \phi_j \, dxdy = 0 \quad \forall j \in \{J_i\}. \tag{18}$$

Although it is customary to write these equations in the form of linear systems of equations by moving all the known quantities to the right hand side, we leave these equations as above for the purpose of comparison. But note that the boundary integral in (17) is a known quantity that can be put on the right hand side since v_h is known on $\partial\Omega$, and that the left hand side of (18) contains the nodal values given by $g(x, y)$ which therefore can also be moved to the right hand side as data.

4 Least-squares Method

In least-squares method, we begin by defining the residuals Ω_T and Δ_T as the integrals of (1) and (2) over each element.

$$\Delta_T = \iint_T \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dxdy = \left(\frac{\partial u^T}{\partial x} + \frac{\partial v^T}{\partial y} \right) S_T \tag{19}$$

$$\Omega_T = \iint_T \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) dxdy = \left(\frac{\partial u^T}{\partial y} - \frac{\partial v^T}{\partial x} \right) S_T \tag{20}$$

where the second equalities are due to the linear approximations of u and v. We then define the norm $\mathcal{F}$ to be minimized by

$$\mathcal{F} = \sum_{T \in \{T\}} F_T = \sum_{T \in \{T\}} \frac{1}{2S_T} \left[\Delta_T^2 + \Omega_T^2 \right] \tag{21}$$

which can be expanded as

$$\begin{aligned} \mathcal{F} = \; & \frac{1}{2} \sum_{T \in \{T\}} \left[\left(\frac{\partial u^T}{\partial x} \right)^2 + \left(\frac{\partial u^T}{\partial y} \right)^2 \right] S_T + \frac{1}{2} \sum_{T \in \{T\}} \left[\left(\frac{\partial v^T}{\partial x} \right)^2 + \left(\frac{\partial v^T}{\partial y} \right)^2 \right] S_T \\ & + \sum_{T \in \{T\}} \left[\frac{\partial u^T}{\partial x} \frac{\partial v^T}{\partial y} - \frac{\partial u^T}{\partial y} \frac{\partial v^T}{\partial x} \right] S_T \end{aligned} \tag{22}$$

or

$$\begin{aligned} \mathcal{F} = \; & \sum_{T \in \{T\}} \frac{1}{2} \iint_T \nabla u^T \cdot \nabla u^T \, dx \, dy + \sum_{T \in \{T\}} \frac{1}{2} \iint_T \nabla v^T \cdot \nabla v^T \, dx \, dy \\ & + \sum_{T \in \{T\}} \left[\frac{\partial u^T}{\partial x} \frac{\partial v^T}{\partial y} - \frac{\partial u^T}{\partial y} \frac{\partial v^T}{\partial x} \right] S_T. \end{aligned} \tag{23}$$

By the relations (11), we have

$$\mathcal{F} = \frac{1}{2} \iint_\Omega \nabla u_h \cdot \nabla u_h \, dx \, dy + \frac{1}{2} \iint_\Omega \nabla v_h \cdot \nabla v_h \, dx \, dy + \sum_{T \in \{T\}} \left[\frac{\partial u^T}{\partial x} \frac{\partial v^T}{\partial y} - \frac{\partial u^T}{\partial y} \frac{\partial v^T}{\partial x} \right] S_T. \tag{24}$$

Now there are two arrangements we can make for the last term. The first option is to use an algebraic identity given by

$$S_T^{\mathcal{H}} = \left[\frac{\partial u^T}{\partial x} \frac{\partial v^T}{\partial y} - \frac{\partial u^T}{\partial y} \frac{\partial v^T}{\partial x} \right] S_T \tag{25}$$

where $S_T^{\mathcal{H}}$ is the area of the image of the element T in the solution space (u, v). This is a discrete analog of

$$du\, dv = (\partial_x u \partial_y v - \partial_y u \partial_x v)\, dx\, dy \tag{26}$$

which holds true exactly for piecewise linear functions. Note that $\partial_x u \partial_y v - \partial_y u \partial_x v$ is always negative. It now follows immediately that the last term reduces to the entire area of the domain in the solution space. Hence we can write

$$\mathcal{F} = \frac{1}{2} \iint_\Omega \nabla u_h \cdot \nabla u_h \, dx\, dy + \frac{1}{2} \iint_\Omega \nabla v_h \cdot \nabla v_h \, dx\, dy + S_{\mathcal{H}} \tag{27}$$

where $S_{\mathcal{H}}$ represents the entire area of the domain in the solution space. It is important to note that $S_{\mathcal{H}}$ involves only the boundary values of u_h and v_h. The second option is to use the relations (11) again. This gives

$$\mathcal{F} = \frac{1}{2} \iint_\Omega \nabla u_h \cdot \nabla u_h \, dx\, dy + \frac{1}{2} \iint_\Omega \nabla v_h \cdot \nabla v_h \, dx\, dy + \iint_\Omega \left[\frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial y} - \frac{\partial u_h}{\partial y} \frac{\partial v_h}{\partial x} \right] dx\, dy. \tag{28}$$

To simplify the last integral, we first use the following identity

$$\frac{\partial u_h}{\partial x} \frac{\partial v_h}{\partial y} - \frac{\partial u_h}{\partial y} \frac{\partial v_h}{\partial x} = \frac{\partial}{\partial x} \left(u_h \frac{\partial v_h}{\partial y} \right) - \frac{\partial}{\partial y} \left(u_h \frac{\partial v_h}{\partial x} \right) \tag{29}$$

to rewrite the integrand, and then use Green's formula to obtain

$$\mathcal{F} = \frac{1}{2} \iint_\Omega \nabla u_h \cdot \nabla u_h \, dx\, dy + \frac{1}{2} \iint_\Omega \nabla v_h \cdot \nabla v_h \, dx\, dy + \oint_{\partial\Omega} u_h \frac{\partial v_h}{\partial s} ds. \tag{30}$$

To find a minimum, we need to solve the following equations

$$\frac{\partial \mathcal{F}}{\partial u_j} = 0 \quad \forall j \in \{J\} \tag{31}$$

$$\frac{\partial \mathcal{F}}{\partial v_j} = 0 \quad \forall j \in \{J_i\} \tag{32}$$

for the nodal unknowns. Now we substitute (30) into the first equation and (27) into the second equation to get

$$\frac{\partial \mathcal{F}}{\partial u_j} = \iint_\Omega \nabla u_h \cdot \nabla \phi_j \, dx\, dy - \oint_{\partial\Omega} \phi_j \frac{\partial v_h}{\partial s} ds = 0 \quad \forall j \in \{J\} \tag{33}$$

$$\frac{\partial \mathcal{F}}{\partial v_j} = \iint_\Omega \nabla v_h \cdot \nabla \phi_j \, dx\, dy = 0 \quad \forall j \in \{J_i\}. \tag{34}$$

These equations are identical to those of finite element method (17) and (18).

5 Concluding Remarks

We have proved the equivalence of the least-squares method for the Cauchy-Riemann system and finite element method for the associated Laplace equations. The resulting linear systems are identical, and therefore have the same unique solution regardress of the choice of the solution algorithm. In particular, if an iterative algorithm is used such as Gauss-Seidel, which can be interpreted also as a variant of the steepest descent method when applied to least-squares method, discrete updates will be the same, within a relaxation factor.

