Analytic Solutions for Some Special MHD Nozzle Flows

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1 Gasdynamic Nozzle Flow

Consider a nozzle with a moderate area variation along its axial direction. Within such a nozzle, the variation of flow properties is almost confined along the x-axis which is taken parallel to the centerline (see Figure 1). Quasi-one-dimensional flow is characterized by the assumption that there are no transverse velocity components and all the remaining variables are functions of x only,

$$p = p(x),\ \rho = \rho(x),\ V_x = u(x), \tag{1}$$

but the area is allowed to vary

$$A = A(x). \tag{2}$$

We then seek to determine the flow properties (1) for a given nozzle shape $A(x)$. The basic equations that we need are

$$\rho u A = const. \tag{3}$$

$$\frac{u^2}{2} + \frac{a^2}{\gamma - 1} = const. \tag{4}$$

$$p/\rho^\gamma = const. \tag{5}$$

where $a = \sqrt{\gamma p/\rho}$. Relating an arbitrary state with the sonic point ($u = a$) via (3), we obtain

$$\rho u A = \rho^* u^* A^* \tag{6}$$

where the sonic condition is indicated by asterisk, which, by introducing the stagnation condition $()_0$, is written in the form

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{u^*}{u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{u^*}{u}. \tag{7}$$

Once we obtain the three ratios on the right hand side in terms of Mach number only, we have a desired result as it then can be solved (numerically perhaps) for Mach number

Figure 1: A Nozzle and the coordinate axes.

for a given A. These quantities can be found from the energy equation (4). The energy conservation between an arbitrary state and the stagnation point is

$$\frac{u^2}{2} + \frac{a^2}{\gamma - 1} = \frac{a_0^2}{\gamma - 1}. \tag{8}$$

Division by a^2 gives

$$\frac{M^2}{2} + \frac{1}{\gamma - 1} = \frac{1}{\gamma - 1}\frac{a_0^2}{a^2}, \tag{9}$$

which by the isentropic relation

$$a_0^2/a^2 = (\rho_0/\rho)^{\gamma-1} \tag{10}$$

yields

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma-1}}. \tag{11}$$

This is one of the desired ratio. Another is derived from taking this to the sonic point,

$$\frac{\rho_0}{\rho^*} = \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma-1}}. \tag{12}$$

Now, consider the energy conservation between an arbitrary state and the sonic point,

$$\frac{u^2}{2} + \frac{a^2}{\gamma - 1} = \frac{\gamma + 1}{2(\gamma - 1)}\, a^{*2}. \tag{13}$$

Division by u^2 and some algebraic maniputlation yields (note that $u^* = a^*$)

$$\frac{u^*}{u} = \sqrt{\frac{1 + \frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}M^2}} \tag{14}$$

which is the final piece we needed. Now, substituting (11), (12), and (14) into (7), we finally obtain

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)} \tag{15}$$

which is the desired result. For a given nozzle shape $A(x)$, this can be solved numerically for Mach number $M(x)$. Once the Mach number is computed, other flow properties follow as they are functions of Mach number only.

Obviously there are two Mach numbers associated with a particular area ratio $\frac{A}{A^*}$: one for subsonic and the other for supersonic flow. A solution in which the flow accelerates from subsonic to supersonic with $M = 1$ precisely at the throat $A = A^*$ can be uniquely determined for a given nozzle. Another solution would be a subsonic flow over the entire nozzle. Yet, another solution is a flow with a normal shock somewhere in the supersonic region. The latter two depend on the exit pressure. See [1] for further details.

2 Aligned MHD Nozzle Flow

Aligned MHD flow is the one for which $\mathbf{B} \parallel \mathbf{V}$. In this case, we have [2], with α being an arbitrary constant

$$\mathbf{B} = \alpha\rho\mathbf{V} \tag{16}$$

which under the quasi-one-dimensional assumption becomes

$$B_x(x) = \alpha\rho u(x). \tag{17}$$

Hence there exists only a streamwise component in the magnetic field. It can be shown that in such a case the streamwise magnetic field component disappears completely in the governing equations for the quasi-one-dimensional flow and therefore the basic equations become identical to those in gasdynamics. Consequently, the solution for such a MHD flow can be obtained simply by computing $B_x(x)$ using (17) after other variables are determined as in the gasdynamics case.

3 Transverse MHD Nozzle Flow

We now consider the case in which $\mathbf{B} \perp \mathbf{V}$, or $\mathbf{B} = (0, 0, B)$. Here, the $z-$direction is taken normal to the plane in which the velocity vector resides. In this case, the basic equations are given by [3]

$$\rho u A = const. \tag{18}$$

$$\frac{u^2}{2} + \frac{a^2}{\gamma-1} + \frac{B^2}{\rho} = const. \tag{19}$$

$$p/\rho^\gamma = const. \tag{20}$$

with

$$\frac{B}{\rho} = const. \tag{21}$$

where $\mathbf{B}$ has been scaled by $\sqrt{\mu_0}$. The difficulty in obtaining area-Mach number relation comes from the extra term in the energy conservation, B^2/ρ. However it can be made look identical to the gasdynamic case when $\gamma = 2$. Suppose $\gamma = 2$, and then write

$$\frac{u^2}{2} + c^2 \quad = \quad const. \tag{22}$$

where

$$c^2 = a^2 + a_A^2 \tag{23}$$

and $a_A = B/\sqrt{\rho}$ is the Alfven speed. Using the stagnation state, we obtain

$$\frac{M^2}{2} + 1 \quad = \quad \left(\frac{c_0}{c}\right)^2. \tag{24}$$

Note that the Mach number, M, is here defined by $M = u/\sqrt{a^2 + a_A^2}$ [1] In order to get the density ratio $\frac{\rho_0}{\rho}$ in terms of Mach number, we must convert the ratio $\frac{c_0}{c}$ to the density ratio. By definition, we have

$$\left(\frac{c_0}{c}\right)^2 = \frac{2p_0/\rho_0 + B_0^2/\rho_0}{2p/\rho + B^2/\rho}, \tag{25}$$

which can be arranged into the form

$$\left(\frac{c_0}{c}\right)^2 = \frac{2(p_0/\rho_0^2) + (B_0/\rho_0)^2}{2(p/\rho^2) + (B/\rho)^2}\left(\frac{\rho}{\rho_0}\right). \tag{26}$$

Owing to (5) with $\gamma = 2$ and (21), this becomes

$$\left(\frac{c_0}{c}\right)^2 = \left(\frac{\rho_0}{\rho}\right). \tag{27}$$

The speed c therefore behaves just like the speed of sound in the gasdynamic case, at least for the relationship with the density ratio (comapre this with (10) for $\gamma = 2$). We then obtain from (22)

$$\frac{\rho_0}{\rho} = 1 + \frac{M^2}{2}, \tag{28}$$

and also

$$\frac{\rho_0}{\rho^*} = \frac{3}{2}. \tag{29}$$

[1] This is the Mach number that characterizes the nozzle flow as it appears in the equation, $\left(\frac{u^2}{a^2+a_A^2} - 1\right)\frac{1}{u}\frac{du}{dx} = \frac{1}{A}\frac{dA}{dx}$ (for any value of γ).

Compare these with (11) and (12). The ratio $\frac{u^*}{u}$ can also be obtained easily from the energy conservation.

$$\frac{u^*}{u} = \sqrt{\frac{1+\frac{M^2}{2}}{\frac{3}{2}M^2}} \tag{30}$$

in the same way in Section 1. Finally, substituting these results into the continuity, we obtain

$$\left(\frac{A}{A^*}\right)^2 = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{u^*}{u} = \frac{1}{M^2}\left[\frac{2}{3}\left(1+\frac{M^2}{2}\right)\right]^3. \tag{31}$$

This is identical to (15) with $\gamma = 2$ except for the definition of the Mach number. This suggests that the flow properties can be determined as in the gasdynamic case with $\gamma = 2$ and then the magnetic field component B can be obtained *a posteriori* by (21), i.e.

$$B(x) = (B_0/\rho_0)\,\rho(x). \tag{32}$$

4 Parallel-Transverse MHD Nozzle Flow

The results from Section 2 and 3 may be combined to produce a flow with $\mathbf{B} = (B_x, 0, B_z)$.

References

[1] Anderson, J. D., *Modern Compressible Flow*, McGraw-Hills, 1990.

[2] Grad, H., Reducible Problems in Magneto-Fluid Dynamic Steady Flows, *Review of Modern Physics*, **32**, No. 4, pp. 830-847, 1960.

[3] Liffman, K. and Siora A., Magnetosonic Jet Flow, *Mon. Not. R. Astron. Soc.*, **290**, pp. 629-635, 1997.

