Norm-Reducing Schemes for Linear Advection on Triangular Grids

Hiroaki Nishikawa

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1 Norm-Reducing Property

We consider solving the linear advection equation

$$u_t + a\partial_x u + b\partial_y u = 0 \tag{1}$$

in a triangulated domain. Define the fluctuation over the triangle T,

$$\phi = \int_T \mathbf{u}_t = \int_T (a\partial_x u + b\partial_y u) \tag{2}$$

Storing the solutions at vertices, and assuming a piecewise linear variation over the triangle with vertices $i = 1, 2, 3$, we obtain

$$\phi = \sum_i k_i u_i \tag{3}$$

where $k_i = \frac{1}{2}(a, b) \cdot \mathbf{n}_i$ and $\mathbf{n}_i$ is the inward normal vector of the edge opposite to the node j. Note that the fluctuation ϕ can be rewritten in many different ways:

$$\begin{aligned} \phi &= k_1(u_1 - u_3) + k_2(u_2 - u_3) && (4) \\ &= \sum_i k_i^+ (u_{out} - u_{in}) && (5) \end{aligned}$$

where $u_{out} = \sum_i k_i^+ u_i / \sum k_i^+$, $u_{in} = \sum_i k_i^- u_i / \sum k_i^-$, $k_i^+ = \max(0, k_i)$, and $k_i^- = \min(0, k_i)$. To update the nodal solutions, we distribute the fluctuation to the nodes in the following form,

$$u_j^{n+1} \longleftarrow u_j^n - \omega\, \phi_i \tag{6}$$

where ω is a positive constant and thus ϕ_i is a fraction of the fluctuation to be distributed to the node j, i.e. $\sum_i \phi_i = \phi$. Various schemes are available to split the fluctuation into appropriate fractions, such as the N scheme, the PSI scheme, the LDA scheme, etc. These schemes are classified by some useful properties such as positivity and linearity-preserving. In this paper, we introduce a yet another property that we call *norm-reducing*. Based on the observation that no schemes can drive all the fluctuations to zero at convergence on triangular grids, we ask if a scheme is reducing the fluctuations. To see this, consider the L_2 norm

$$\mathcal{F} = \frac{1}{2} \sum_T \phi^T Q^T \phi^T \tag{7}$$

where Q^T is a positive weight. Its variation due to a small change in the solution at node j, δu_j, is given by

$$\delta \mathcal{F} = \sum_{\{T_j\}} \delta \mathcal{F}^T = \sum_{\{T_j\}} k_j^T \phi^T \delta u_j \tag{8}$$

where $\{T_j\}$ is a group of triangles that share the node j, and we have taken $Q^T = 1$ for simplicity. If the small change δu_j is caused by a distribution scheme in the form (6), this can be written as

$$\delta \mathcal{F} = -\omega \sum_{\{T_j\}} k_j^T \, \phi^T \, \phi_j^T. \tag{9}$$

Then, the scheme is norm-reducing if

$$\sum_{\{T_j\}} k_j^T \, \phi^T \, \phi_j^T > 0 \tag{10}$$

which guarantees that

$$\delta \mathcal{F} < 0. \tag{11}$$

As in the positivity property, it is often easier to consider its local counterpart although this typically yields stronger conditions. Let us consider a local contribution from triangle $T \in \{T_j\}$, dropping the superscript T,

$$\delta \mathcal{F}^T = -\omega \, k_j \, \phi \, \phi_j. \tag{12}$$

A scheme is then called locally norm-reducing within a triangle T if

$$k_j \, \phi \, \phi_j > 0 \tag{13}$$

which guarantees that

$$\delta \mathcal{F}^T < 0. \tag{14}$$

2 Norm-Reducing Schemes

It is easy to prove that *sending the whole fluctuation to the downwind node guarantees norm-reducing.* This is simply because in such a case $\phi_j = \phi$ for $k_j > 0$, i.e.

$$k_j \, \phi \, \phi_j = k_j \, (\phi)^2 > 0 \tag{15}$$

and thus norm-reducing. Note that a scheme is norm-reducing whenever it sends the whole fluctuation to the node for which $k_i > 0$, and this includes upwind schemes in the one-target situation. It is interesting that this shows that sending the whole fluctuation to upwind nodes for which $k_j < 0$ yields a norm-increasing scheme, thus implying instability.

From here on, we consider the two-target case only and assume that $k_1 > 0$, $k_2 > 0$, and $k_3 < 0$ and that $\phi_3 = 0$. We then consider the variation of the norm due to δu_1 and δu_2,

$$\delta \mathcal{F}^T = -\omega\, k_1\, \phi\, \phi_1 - \omega\, k_2\, \phi\, \phi_2 = -\omega\, \phi\, (k_1\, \phi_1 + k_2\, \phi_2) \tag{16}$$

which can be expanded, using

$$\phi_1 + \phi_2 = \phi, \tag{17}$$

as

$$\begin{aligned} \delta \mathcal{F}^T &= -\omega\, (\phi_1 + \phi_2)\, (k_1\, \phi_1 + k_2\, \phi_2) && (18) \\ &= -\omega\, \left\{ k_1 \phi_1^2 + k_2 \phi_2^2 + (k_1 + k_2)\, \phi_1 \phi_2 \right\} && (19) \\ &= -\omega \left(\sqrt{k_1} \phi_1 + \sqrt{k_2} \phi_2 \right)^2 - \omega \left(\sqrt{k_1} - \sqrt{k_2} \right)^2 \phi_1 \phi_2. && (20) \end{aligned}$$

It is obvious from this that the norm may increase if $\phi_1 \phi_2$ is negative. Also note that a scheme is norm-reducing if $k_1 = k_2$.

The condition for norm-reducing can be expressed in various forms. An example is

$$k_1 \phi_1^2 + k_2 \phi_2^2 + (k_1 + k_2)\, \phi_1 \phi_2 > 0 \tag{21}$$

or

$$\phi_1 \phi_2 > -\frac{k_1 \phi_1^2 + k_2 \phi_2^2}{k_1 + k_2} \tag{22}$$

This means that the product $\phi_1 \phi_2$ does not have to be positive but must be greater than a certain negative value.

We first consider the LDA scheme, which is in the two-target case given by

$$\begin{aligned} \phi_1 &= \frac{k_1}{k_1 + k_2}\, \phi && (23) \\ \phi_2 &= \frac{k_2}{k_1 + k_2}\, \phi. && (24) \end{aligned}$$

Substituting these into (16) gives

$$\delta \mathcal{F}^T = -\omega\, \phi^2\, \frac{k_1^2 + k_2^2}{k_1 + k_2} < 0 \tag{25}$$

Therefore, the LDA scheme is norm-reducing. This is an obvious result because the LDA scheme may be obtained by applying an upwind constraint on the least-squares minimization scheme. The N scheme is given by

$$\begin{aligned} \phi_1 &= k_1\, (u_1 - u_3) && (26) \\ \phi_2 &= k_2\, (u_2 - u_3)\,. && (27) \end{aligned}$$

The product of these two terms can be negative, and therefore the N scheme is not norm-reducing when it is so. The PSI scheme is a limited-version of the N scheme, and it differs from the N scheme in that it sends the whole fluctuation to the node with $k_i > 0$ and $\phi_i/\phi > 0$ when $\phi_1\phi_2 < 0$. This makes the PSI scheme fully norm-reducing.

The condition for norm-reducing may be expressed in terms of distribution coefficients. From (16), we have

$$\delta\mathcal{F}^T = -\omega\,\phi^2\,(k_1\,\beta_1 + k_2\,\beta_2) \tag{28}$$

where $\beta_i = \phi_i/\phi$, and therefore a scheme is norm-reducing if

$$k_1\,\beta_1 + k_2\,\beta_2 > 0 \tag{29}$$

which can be generalized as

$$\sum_i k_i\,\beta_i > 0 \tag{30}$$

Consider a scheme in the form,

$$\beta_i = \frac{1}{3} + Ck_i \tag{31}$$

where C is a positive constant. This includes the Lax-Wendroff scheme with

$$C = \frac{\Delta t}{2S_T} \tag{32}$$

and also the SUPG scheme with

$$C = \frac{h}{2|\lambda|S_T}. \tag{33}$$

This type of scheme is norm-reducing because

$$\begin{aligned}\sum_i k_i\,\beta_i &= \sum_i k_i\left(\frac{1}{3} + Ck_i\right) && (34)\\ &= \frac{1}{3}\sum_i k_i + C\sum_i k_i^2 && (35)\\ &= C\sum_i k_i^2 > 0 && (36)\end{aligned}$$