On Viscous Scheme for Cell-Centered Method

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November 4, 2015

In finite-volume discretizations of diffusion terms, it is the evaluation of the normal derivative at a face that characterizes the diffusion scheme (See, e.g., Refs.[4, 5]). Here, we consider a particular formula in Refs.[1, 2] for second-order discretizations, derive an alternative form, discuss its relation to a known diffusion scheme, and suggest an improved version that can achieve fourth-order accuracy on regular grids.

1 Normal Derivative Formula

Consider the finite-difference approximation to a normal derivative as described in Refs.[1, 2]:

$$\frac{\partial \phi}{\partial n} = \frac{\phi_{ifn2} - \phi_{ifn1}}{d_f}, \tag{1}$$

where d_f is the distance between two points forming a line perpendicular to the face, and

$$\phi_{ifn1} = \phi_{icv1} + \nabla\phi_{icv1} \cdot (\mathbf{x}_{ifn1} - \mathbf{x}_{icn1}), \quad \phi_{ifn2} = \phi_{icv2} + \nabla\phi_{icv2} \cdot (\mathbf{x}_{ifn2} - \mathbf{x}_{icn2}). \tag{2}$$

See Figure 1 for notations (or see Figure 2.1 in Ref.[2]).

I noticed that this formula has an alternative form. Let $\mathbf{x}_m$ be the midpoint coordinate of the face. Then, the value ϕ_{ifn1} can be written as

$$\begin{aligned}
\phi_{ifn1} &= \phi_{icv1} + \nabla\phi_{icv1} \cdot (\mathbf{x}_{ifn1} - \mathbf{x}_{icn1}) \\
&= \phi_{icv1} + \nabla\phi_{icv1} \cdot (\mathbf{x}_{m} - \mathbf{x}_{icn1}) + \nabla\phi_{icv1} \cdot (\mathbf{x}_{ifn1} - \mathbf{x}_{m}) \\
&= \nabla\phi_{icv1} \cdot (\mathbf{x}_{ifn1} - \mathbf{x}_{m}) + \phi_L \\
&= \nabla\phi_{icv1}(-\hat{\mathbf{n}}) d_{f1} + \phi_L
\end{aligned} \tag{3}$$

and similarly for ϕ_{ifn2}

$$\begin{aligned}
\phi_{ifn2} &= \phi_{icv2} + \nabla\phi_{icv2} \cdot (\mathbf{x}_{ifn2} - \mathbf{x}_{icn2}) \\
&= \phi_{icv2} + \nabla\phi_{icv2} \cdot (\mathbf{x}_{m} - \mathbf{x}_{icn2}) + \nabla\phi_{icv1} \cdot (\mathbf{x}_{ifn2} - \mathbf{x}_{m}) \\
&= \nabla\phi_{icv2} \cdot (\mathbf{x}_{ifn2} - \mathbf{x}_{m}) + \phi_R \\
&= \nabla\phi_{icv1} \hat{\mathbf{n}} d_{f2} + \phi_R
\end{aligned} \tag{4}$$

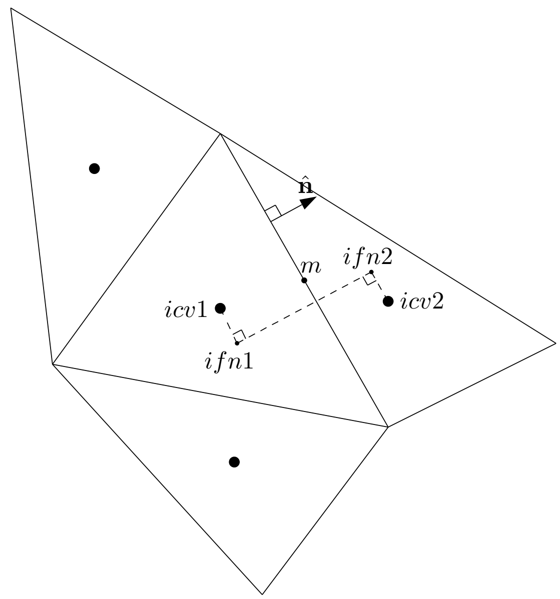

Figure 1: Stencil for a cell-centered diffusion scheme. The face-midpoint is denoted by m. The unit face-normal vector is denoted by $\hat{\mathbf{n}}$. The centroids are denoted by $icv1$ and $icv2$. The points $ifn1$ and $ifn2$ form a line perpendicular to the face.

where $\hat{\mathbf{n}}$ denotes the unit face normal vector, d_{f1} and d_{f2} denote the distances between the face midpoint and the point $ifn1$ and $ifn2$, respectively (i.e., $d_{f1} + d_{f2} = d_f$), and we have introduced the notations:

$$\phi_L = \phi_{icv1} + \nabla\phi_{icv1} \cdot (\mathbf{x}_m - \mathbf{x}_{icn1}), \quad \phi_R = \phi_{icv2} + \nabla\phi_{icv2} \cdot (\mathbf{x}_m - \mathbf{x}_{icn2}). \tag{5}$$

Note that ϕ_L and ϕ_R are the values of ϕ linearly extrapolated from the cell centers to the face midpoint. Substituting Equations (3) and (4) into Equation (1), we obtain

$$\frac{\partial\phi}{\partial n} = \frac{\nabla\phi_{icv1} d_{f1} + \nabla\phi_{icv2} d_{f2}}{d_{f1} + d_{f2}} \cdot \hat{\mathbf{n}} + \frac{1}{d_f}\left(\phi_R - \phi_L\right). \tag{6}$$

It shows that the normal derivative formula has two parts: distance-weighted LSQ gradient projected along the face normal vector and a high-frequency damping term $\frac{1}{d_f}\left(\phi_R - \phi_L\right)$, and the damping coefficient is given by $1/d_f$.

2 Alpha-Damping Scheme

In Ref.[3] (See also Refs.[4, 5]), a new principle for constructing viscous schemes is proposed for various discretization methods (FV, RD, DG, SV). The one constructed for FV methods is called the alpha-damping scheme:

$$\frac{\partial \phi}{\partial n} = \frac{\nabla \phi_{icv1} + \nabla \phi_{icv2}}{2} \cdot \hat{\mathbf{n}} + \frac{\alpha}{|\mathbf{e} \cdot \hat{\mathbf{n}}|} \left(\phi_R - \phi_L \right), \tag{7}$$

where α is a parameter, and $\mathbf{e} = \mathbf{x}_{icn2} - \mathbf{x}_{icn1}$. The first term is called the consistent part since it consistently approximates the normal derivative. The second term is called the damping term since it has an effect of damping high-frequency errors [3]. A recommended value for α is 4/3, which gives fourth-order accuracy on regular grids[3]. On a uniformly-spaced grid, the resulting scheme is equivalent to the fourth-order central finite-difference scheme.

The quantity $|\mathbf{e} \cdot \hat{\mathbf{n}}|$ is a measure of the grid skewness: $|\mathbf{e} \cdot \hat{\mathbf{n}}| \to 0$ as the skewness increases, i.e., as the angle formed by $\mathbf{e}$ and $\mathbf{n}$ approaches 90 degrees. It amplifies the effect of high-frequency damping, thus making the scheme robust on highly-skewed grids[3]. The scheme has been tested and analyzed by Jalali et. al. in Ref.[6], and concluded as one of the most robust viscous schemes (with $\alpha = 4/3$) for fully unstructured grids.

Let us compare the alpha-damping scheme and the formula (1) in the form (6). First, the alpha-damping scheme has the consistent part the arithmetic average of the LSQ gradients projected along $\hat{\mathbf{n}}$ while the formula (1) defines the consistent part as the distance-averaged LSQ gradients along $\hat{\mathbf{n}}$. Variations in the consistent part have been studied by Jalali et. al. in Ref.[6], but no significant differences were observed. It is the damping term that has a major impact on viscous schemes. For the damping term, we see form Equation (6) that the formula (1) is equally robust since d_f in the denominator is exactly the skewness measure:

$$d_f = |\mathbf{e} \cdot \hat{\mathbf{n}}|. \tag{8}$$

It is straightforward to adapt the parameter α in the formula (1):

$$\frac{\partial \phi}{\partial n} = \frac{\nabla \phi_{icv1} d_{f1} + \nabla \phi_{icv2} d_{f2}}{d_{f1} + d_{f2}} \cdot \hat{\mathbf{n}} + \frac{\alpha}{d_f} \left(\phi_R - \phi_L \right). \tag{9}$$

The optimal value of α may depend on the consistent part since $\alpha = 4/3$ was derived with the arithmetic averaged consistent term. A different optimal value may exist for the distance-weighted consistent term. Or we can employ the arithmetic average:

$$\frac{\partial \phi}{\partial n} = \frac{\nabla \phi_{icv1} + \nabla \phi_{icv2}}{2} \cdot \hat{\mathbf{n}} + \frac{\alpha}{d_f} \left(\phi_R - \phi_L \right). \tag{10}$$

Then, $\alpha = 4/3$ is an optimal value. Note that the coefficient of the damping term can be freely adjusted without affecting the order of accuracy of the scheme because $(\phi_R - \phi_L)$ has an order property (i.e., it goes to zero as the grid is refined) just like the dissipation term of adevective schemes.

Another interesting variant would be the inverse-distance-weighted consistent term:

$$\frac{\partial \phi}{\partial n} = \frac{\nabla \phi_{icv1} \frac{1}{d_{f1}} + \nabla \phi_{icv2} \frac{1}{d_{f2}}}{\frac{1}{d_{f1}} + \frac{1}{d_{f2}}} \cdot \hat{\mathbf{n}} + \frac{\alpha}{d_f} \left(\phi_R - \phi_L \right), \tag{11}$$

which emphasizes the cell closer to the face midpoint.

In all variations in the consistent term, the scheme reduces to the arithmetic average if $d_{f1} = d_{f2}$, which includes uniform grids. And therefore, all schemes will reduce to the fourth-order central scheme with $\alpha = 4/3$. This may be an attractive feature for DNS/LES calculations with relatively regular grids.

References

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