# RKDG Schemes for the Divergence-Free MHD Method on Cartesian Meshes

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**Abstract**

In this note, we explicitly write out the scheme resulting from the RKDG formulation for three-dimensional MHD equations with the divergence-free technique of Balsara.

## 1 Geometry of Computational Cells

We consider Cartesian meshes in this note, so that a computational cell of interest here is a regular cube with its length in each dimension denoted by $\Delta x, \Delta y, \Delta z$. The coordinates used throughout this note are local:

$$-\frac{\Delta x}{2} \leq x \leq \frac{\Delta x}{2}, \;\; -\frac{\Delta y}{2} \leq y \leq \frac{\Delta y}{2}, \;\; -\frac{\Delta z}{2} \leq z \leq \frac{\Delta z}{2}. \tag{1}$$

Of course, because it is a cube, we have $\Delta x = \Delta y = \Delta z$. The volume of the cell is $\Delta V = \Delta x \Delta y \Delta z$.

## 2 $P^1$ RKDG (2nd Order Scheme)

### 2.1 Basis Functions

Within each cell, for each cell-centered variables, we define a piecewise linear variation in the following form.

$$u_h(x,y,z,t) \;\; = \;\; u_0(t)\phi_0 + u_x(t)\phi_1 + u_y(t)\phi_2 + u_z(t)\phi_3 \tag{2}$$

where

$$\phi_0 = 1, \;\; \phi_1 = \frac{x}{\Delta x}, \;\; \phi_2 = \frac{y}{\Delta y}, \;\; \phi_3 = \frac{z}{\Delta z}, \tag{3}$$

and the variables to be evolved are

$$u_0(t), \;\; u_x(t), \;\; u_y(t), \;\; u_z(t). \tag{4}$$

Note that the basis functions are orthogonal, and therefore we have

$$\int_V \phi_i \phi_j \, dV = \Delta V \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{1}{12} & 0 \\ 0 & 0 & 0 & \frac{1}{12} \end{bmatrix}. \tag{5}$$

The gradient of each basis is given by

$$\mathrm{grad}\,\phi_0 \;\; = \;\; 0 \tag{6}$$

$$\operatorname{grad}\phi_1 = \left(\frac{1}{\Delta x}, 0, 0\right) \tag{7}$$

$$\operatorname{grad}\phi_2 = \left(0, \frac{1}{\Delta y}, 0\right) \tag{8}$$

$$\operatorname{grad}\phi_3 = \left(0, 0, \frac{1}{\Delta z}\right). \tag{9}$$

$$\tag{10}$$

In the divergence-free method, we carry the normal component of the magnetic field on the faces. Hence, within each face also, we define a piecewise linear variation in a similar form.

$$B^x\left(x = \frac{\Delta x}{2}, y, z, t\right) = B_0^x(t)\psi_0 + B_y^x(t)\psi_1 + B_z^x(t)\psi_2 \tag{11}$$

where

$$\psi_0 = 1, \;\; \psi_1 = \frac{y}{\Delta y}, \;\; \psi_2 = \frac{z}{\Delta z} \tag{12}$$

and the variables to be evolved are

$$B_0^x(t), \;\; B_y^x(t), \;\; B_z^x(t) \tag{13}$$

Note that the basis functions are again orthogonal, and therefore we have

$$\int_S \psi_i \psi_j \, dS = \Delta y \Delta z \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}. \tag{14}$$

The gradient of each basis is given by

$$\operatorname{grad}\psi_0 = 0 \tag{15}$$

$$\operatorname{grad}\psi_1 = \left(0, \frac{1}{\Delta y}, 0\right) \tag{16}$$

$$\operatorname{grad}\psi_2 = \left(0, 0, \frac{1}{\Delta z}\right) \tag{17}$$

$$\tag{18}$$

The variations of y-component and z-component, ($B^y$ and $B^z$) are defined analogously (a matter of a simple cyclic premutation among $x, y, z$).

## 2.2 The Conservation Law

Consider the three-dimensional MHD equations in the conservative form.

$$\partial_t \mathbf{U} + \operatorname{div}\mathbf{F} = 0 \tag{19}$$

where $\mathbf{U}$ and $\mathbf{F}$ denote a vector of conservative variables and the associated flux tensor $\mathbf{F} = (F, G, H)$. Let $v$ be a test function. Then, we define its weak form by

$$\int_V \partial_t \mathbf{U}\, v \, dV + \int_V \operatorname{div}\mathbf{F}\, v \, dV = 0 \tag{20}$$

which becomes

$$\int_V \partial_t \mathbf{U}\, v \, dV + \oint_{\partial V} v \mathbf{F}_n - \int_V \mathbf{F} \operatorname{grad} v \, dV = 0 \tag{21}$$

where $\mathbf{F}_n$ is the normal flux vector.

Now, introducing the finite-element approximation, and choosing the test function to be one of the basis function, we obtain

$$\int_V \partial_t \mathbf{U}_h \, \phi_j \, dV + \oint_{\partial V} \phi_j \mathbf{F}_n \, dS - \int_V \mathbf{F} \, \mathrm{grad}\phi_j \, dV = 0. \tag{22}$$

Note that the integrals are restricted to a particular cell since basis functions are discontinuous across the cell boundary. By orthogonality of the basis functions, the equations for each variable are completely decoupled as follows.

$$\frac{d\mathbf{U}_0}{dt} = -\frac{1}{\Delta V} \oint_{\partial V} \mathbf{F}_n dS \tag{23}$$

$$\frac{d\mathbf{U}_x}{dt} = -\frac{12}{\Delta V} \left( \oint_{\partial V} \phi_1(x) \mathbf{F}_n \, dS - \frac{1}{\Delta x} \int_V F \, dV \right) \tag{24}$$

$$\frac{d\mathbf{U}_y}{dt} = -\frac{12}{\Delta V} \left( \oint_{\partial V} \phi_2(y) \mathbf{F}_n \, dS - \frac{1}{\Delta y} \int_V G \, dV \right) \tag{25}$$

$$\frac{d\mathbf{U}_z}{dt} = -\frac{12}{\Delta V} \left( \oint_{\partial V} \phi_1(z) \mathbf{F}_n \, dS - \frac{1}{\Delta x} \int_V H \, dV \right) \tag{26}$$

Made specific to a Cartesian cell, these equations are written

$$\begin{aligned} \frac{d\mathbf{U}_0}{dt} &+ \frac{1}{\Delta x} \left[ \int F \left( x = \frac{\Delta x}{2}, y, z \right) dydz - \int F \left( x = -\frac{\Delta x}{2}, y, z \right) dydz \right] \frac{1}{\Delta y \Delta z} \\ &+ \frac{1}{\Delta y} \left[ \int G \left( x, y = \frac{\Delta y}{2}, z \right) dxdz - \int G \left( x, y = -\frac{\Delta y}{2}, z \right) dxdz \right] \frac{1}{\Delta x \Delta z} \\ &+ \frac{1}{\Delta z} \left[ \int H \left( x, y, z = \frac{\Delta z}{2} \right) dxdy - \int H \left( x, y, z = -\frac{\Delta z}{2} \right) dxdy \right] \frac{1}{\Delta x \Delta y} = 0 \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{1}{12}\frac{d\mathbf{U}_x}{dt} &+ \frac{1}{\Delta x} \left[ \int F \left( x = \frac{\Delta x}{2}, y, z \right) \left( \frac{1}{2} \right) dydz - \int F \left( x = -\frac{\Delta x}{2}, y, z \right) \left( -\frac{1}{2} \right) dydz \right] \frac{1}{\Delta y \Delta z} \\ &+ \frac{1}{\Delta y} \left[ \int G \left( x, y = \frac{\Delta y}{2}, z \right) \phi_1(x) \, dxdz - \int G \left( x, y = -\frac{\Delta y}{2}, z \right) \phi_1(x) \, dxdz \right] \frac{1}{\Delta x \Delta z} \\ &+ \frac{1}{\Delta z} \left[ \int H \left( x, y, z = \frac{\Delta z}{2} \right) \phi_1(x) \, dxdy - \int H \left( x, y, z = -\frac{\Delta z}{2} \right) \phi_1(x) \, dxdy \right] \frac{1}{\Delta x \Delta y} \\ &- \frac{1}{\Delta x} \left[ \int_V F(x, y, z) \, dxdydz \right] \frac{1}{\Delta x \Delta y \Delta z} = 0 \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{1}{12}\frac{d\mathbf{U}_y}{dt} &+ \frac{1}{\Delta x} \left[ \int F \left( x = \frac{\Delta x}{2}, y, z \right) \phi_2(y) \, dydz - \int F \left( x = -\frac{\Delta x}{2}, y, z \right) \phi_2(y) \, dydz \right] \frac{1}{\Delta y \Delta z} \\ &+ \frac{1}{\Delta y} \left[ \int G \left( x, y = \frac{\Delta y}{2}, z \right) \left( \frac{1}{2} \right) dxdz - \int G \left( x, y = -\frac{\Delta y}{2}, z \right) \left( -\frac{1}{2} \right) dxdz \right] \frac{1}{\Delta x \Delta z} \\ &+ \frac{1}{\Delta z} \left[ \int H \left( x, y, z = \frac{\Delta z}{2} \right) \phi_2(y) \, dxdy - \int H \left( x, y, z = -\frac{\Delta z}{2} \right) \phi_2(y) \, dxdy \right] \frac{1}{\Delta x \Delta y} \\ &- \frac{1}{\Delta y} \left[ \int_V G(x, y, z) \, dxdydz \right] \frac{1}{\Delta x \Delta y \Delta z} = 0 \end{aligned} \tag{29}$$

$$\frac{1}{12}\frac{d\mathbf{U}_z}{dt} + \frac{1}{\Delta x} \left[ \int F \left( x = \frac{\Delta x}{2}, y, z \right) \phi_3(z) \, dydz - \int F \left( x = -\frac{\Delta x}{2}, y, z \right) \phi_3(z) \, dydz \right] \frac{1}{\Delta y \Delta z} \tag{30}$$

$$
\begin{aligned}
&+ \frac{1}{\Delta y}\left[\int G\left(x, y=\frac{\Delta y}{2}, z\right)\phi_3(z)\,dxdz - \int G\left(x, y=-\frac{\Delta y}{2}, z\right)\phi_3(z)\,dxdz\right]\frac{1}{\Delta x \Delta z} \\
&+ \frac{1}{\Delta z}\left[\int H\left(x, y, z=\frac{\Delta z}{2}\right)\left(\frac{1}{2}\right)\,dxdy - \int H\left(x, y, z=-\frac{\Delta z}{2}\right)\left(-\frac{1}{2}\right)\,dxdy\right]\frac{1}{\Delta x \Delta y} \\
&- \frac{1}{\Delta z}\left[\int_V H\left(x, y, z\right)dxdydz\right]\frac{1}{\Delta x \Delta y \Delta z} = 0
\end{aligned}
$$

## 2.3 The Stokes' Law

Now consider the Stokes' Law

$$\partial_t \mathbf{B} + \operatorname{curl} \mathbf{E} = 0 \tag{31}$$

where $\mathbf{B} = (B^x, B^y, B^z)$ and $\mathbf{E} = (E_x, E_y, E_z)$. Its weak form is given by

$$\frac{d}{dt}\int_{\partial V} v\mathbf{B}\cdot d\mathbf{S} + \int_{\partial V} v \operatorname{curl}\mathbf{E}\cdot d\mathbf{S} = 0 \tag{32}$$

which becomes

$$\frac{d}{dt}\int_{\partial V} v\mathbf{B}\cdot d\mathbf{S} + \int_{\partial V} \operatorname{curl}(v\,\mathbf{E})\cdot d\mathbf{S} - \int_{\partial V}(\operatorname{grad} v \times \mathbf{E})\cdot d\mathbf{S} = 0. \tag{33}$$

Introducing the finite-element approximation, and choosing the test function to be one of the basis function, we obtain

$$\int_V \partial_t \mathbf{U}_h\,\phi_j\,dV + \oint_{\partial S}\phi_j \mathbf{E}\cdot d\ell - \int_S(\operatorname{grad}\phi_j \times \mathbf{E})\cdot d\mathbf{S} = 0. \tag{34}$$

Note again that the integrals are restricted to a particular cell face since basis functions are discontinuous across the cell-face boundary. By orthogonality of the basis functions, the equations for each variable are completely decoupled as follows. Over the face at $x = \frac{\Delta x}{2}$

$$\frac{dB_0^x}{dt} = -\frac{1}{\Delta S}\oint_{\partial S}\mathbf{E}\cdot d\mathbf{l} \tag{35}$$

$$\frac{dB_y^x}{dt} = -\frac{12}{\Delta S}\left(\oint_{\partial S}\psi_1(y)\mathbf{E}\cdot d\mathbf{l} - \frac{1}{\Delta y}\int_S E_z dS\right) \tag{36}$$

$$\frac{dB_z^x}{dt} = -\frac{12}{\Delta S}\left(\oint_{\partial S}\psi_2(z)\mathbf{E}\cdot d\mathbf{l} + \frac{1}{\Delta z}\int_S E_y dS\right) \tag{37}$$

Similarly, at $y = \frac{\Delta y}{2}$

$$\frac{dB_0^y}{dt} = -\frac{1}{\Delta S}\oint_{\partial S}\mathbf{E}\cdot d\mathbf{l} \tag{38}$$

$$\frac{dB_z^y}{dt} = -\frac{12}{\Delta S}\left(\oint_{\partial S}\psi_1(z)\mathbf{E}\cdot d\mathbf{l} - \frac{1}{\Delta z}\int_S E_x dS\right) \tag{39}$$

$$\frac{dB_x^y}{dt} = -\frac{12}{\Delta S}\left(\oint_{\partial S}\psi_2(x)\mathbf{E}\cdot d\mathbf{l} + \frac{1}{\Delta x}\int_S E_z dS\right) \tag{40}$$

and at $z = \frac{\Delta z}{2}$

$$\frac{dB_0^z}{dt} = -\frac{1}{\Delta S}\oint_{\partial S}\mathbf{E}\cdot d\mathbf{l} \tag{41}$$

$$\frac{dB_x^z}{dt} = -\frac{12}{\Delta S}\left(\oint_{\partial S}\psi_1(x)\mathbf{E}\cdot d\mathbf{l} - \frac{1}{\Delta x}\int_S E_y dS\right) \tag{42}$$

$$\frac{dB_y^z}{dt} = -\frac{12}{\Delta S}\left(\oint_{\partial S}\psi_2(y)\mathbf{E}\cdot d\mathbf{l} + \frac{1}{\Delta y}\int_S E_x dS\right) \tag{43}$$

Made more specific to a Cartesian cell, these equations are written, at $x = \frac{\Delta x}{2}$,

$$\begin{aligned} \frac{dB_0^x}{dt} &+ \frac{1}{\Delta y \Delta z}\left\{\int E_z\left(y=\frac{\Delta y}{2}, z\right) dz - \int E_z\left(y=-\frac{\Delta y}{2}, z\right) dz \right. \\ &\left. - \int E_y\left(y, z=\frac{\Delta z}{2}\right) dy + \int E_y\left(y, z=-\frac{\Delta z}{2}\right) dy\right\} = 0 \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{1}{12}\frac{dB_y^x}{dt} &+ \frac{1}{\Delta y \Delta z}\left\{\int E_z\left(y=\frac{\Delta y}{2}, z\right)\left(\frac{1}{2}\right) dz - \int E_z\left(y=-\frac{\Delta y}{2}, z\right)\left(-\frac{1}{2}\right) dz \right. \\ &\left. - \int E_y\left(y, z=\frac{\Delta z}{2}\right)\psi_1(y)\, dy + \int E_y\left(y, z=-\frac{\Delta z}{2}\right)\psi_1(y)\, dy\right\} \\ &- \frac{1}{\Delta y}\left[\int_S E_z dydz\right]\frac{1}{\Delta y \Delta z} = 0 \end{aligned} \tag{45}$$

$$\begin{aligned} \frac{1}{12}\frac{dB_z^x}{dt} &+ \frac{1}{\Delta y \Delta z}\left\{\int E_z\left(y=\frac{\Delta y}{2}, z\right)\psi_2(z)\, dz - \int E_z\left(y=-\frac{\Delta y}{2}, z\right)\psi_2(z)\, dz \right. \\ &\left. - \int E_y\left(y, z=\frac{\Delta z}{2}\right)\left(\frac{1}{2}\right) dy + \int E_y\left(y, z=-\frac{\Delta z}{2}\right)\left(-\frac{1}{2}\right) dy\right\} \\ &+ \frac{1}{\Delta z}\left[\int_S E_y dydz\right]\frac{1}{\Delta y \Delta z} = 0 \end{aligned} \tag{46}$$

Similarly, at $y = \frac{\Delta y}{2}$,

$$\begin{aligned} \frac{dB_0^y}{dt} &+ \frac{1}{\Delta z \Delta x}\left\{\int E_x\left(z=\frac{\Delta z}{2}, x\right) dx - \int E_x\left(z=-\frac{\Delta z}{2}, x\right) dx \right. \\ &\left. - \int E_z\left(z, x=\frac{\Delta x}{2}\right) dz + \int E_z\left(z, x=-\frac{\Delta x}{2}\right) dz\right\} = 0 \end{aligned} \tag{47}$$

$$\begin{aligned} \frac{1}{12}\frac{dB_z^y}{dt} &+ \frac{1}{\Delta z \Delta x}\left\{\int E_x\left(z=\frac{\Delta z}{2}, x\right)\left(\frac{1}{2}\right) dx - \int E_x\left(z=-\frac{\Delta z}{2}, x\right)\left(-\frac{1}{2}\right) dx \right. \\ &\left. - \int E_z\left(z, x=\frac{\Delta x}{2}\right)\psi_1(z)\, dz + \int E_z\left(z, x=-\frac{\Delta x}{2}\right)\psi_1(z)\, dz\right\} \\ &- \frac{1}{\Delta z}\left[\int_S E_x dzdx\right]\frac{1}{\Delta z \Delta x} = 0 \end{aligned} \tag{48}$$

$$\begin{aligned} \frac{1}{12}\frac{dB_x^y}{dt} &+ \frac{1}{\Delta z \Delta x}\left\{\int E_x\left(z=\frac{\Delta z}{2}, x\right)\psi_2(x)\, dx - \int E_x\left(z=-\frac{\Delta z}{2}, x\right)\psi_2(x)\, dx \right. \\ &\left. - \int E_z\left(z, x=\frac{\Delta x}{2}\right)\left(\frac{1}{2}\right) dz + \int E_z\left(z, x=-\frac{\Delta x}{2}\right)\left(-\frac{1}{2}\right) dz\right\} \\ &+ \frac{1}{\Delta x}\left[\int_S E_z dzdx\right]\frac{1}{\Delta z \Delta x} = 0 \end{aligned} \tag{49}$$

Similarly, at $z = \frac{\Delta z}{2}$,

$$\begin{aligned} \frac{dB_0^z}{dt} &+ \frac{1}{\Delta x \Delta y}\left\{\int E_y\left(x=\frac{\Delta x}{2}, y\right) dy - \int E_y\left(x=-\frac{\Delta x}{2}, y\right) dy \right. \\ &\left. - \int E_x\left(x, y=\frac{\Delta y}{2}\right) dx + \int E_x\left(x, y=-\frac{\Delta y}{2}\right) dx\right\} = 0 \end{aligned} \tag{50}$$

$$
\begin{aligned}
\frac{1}{12}\frac{dB_x^z}{dt} \quad + \quad & \frac{1}{\Delta x \Delta y}\left\{\int E_y\left(x=\frac{\Delta x}{2},y\right)\left(\frac{1}{2}\right)dy - \int E_y\left(x=-\frac{\Delta x}{2},y\right)\left(-\frac{1}{2}\right)dy \right. \\
& \left. - \int E_x\left(x,y=\frac{\Delta y}{2}\right)\psi_1(x)\,dx + \int E_x\left(x,y=-\frac{\Delta y}{2}\right)\psi_1(x)\,dx \right\} \\
- \quad & \frac{1}{\Delta x}\left[\int_S E_y dx dy\right]\frac{1}{\Delta x \Delta y} = 0
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\frac{1}{12}\frac{dB_y^z}{dt} \quad + \quad & \frac{1}{\Delta x \Delta y}\left\{\int E_y\left(x=\frac{\Delta x}{2},y\right)\psi_2(y)\,dy - \int E_y\left(x=-\frac{\Delta x}{2},y\right)\psi_2(y)\,dy \right. \\
& \left. - \int E_x\left(x,y=\frac{\Delta y}{2}\right)\left(\frac{1}{2}\right)dx + \int E_x\left(x,y=-\frac{\Delta y}{2}\right)\left(-\frac{1}{2}\right)dx \right\} \\
+ \quad & \frac{1}{\Delta y}\left[\int_S E_x dx dy\right]\frac{1}{\Delta x \Delta y} = 0.
\end{aligned} \tag{52}
$$

# 3 $P^2$ RKDG (3rd Order Scheme)

## 3.1 Basis Functions

Within each cell, we now define a piecewise quadratic variation in the following form.

$$
\begin{aligned}
u_h(x,y,z,t) \quad = \quad & u_0(t)\phi_0 + u_x(t)\phi_1 + u_y(t)\phi_2 + u_z(t)\phi_3 + u_{xx}(t)\phi_4 + u_{xy}(t)\phi_5 \\
& + u_{yy}(t)\phi_6 + u_{yz}(t)\phi_7 + u_{zz}(t)\phi_8 + u_{zx}(t)\phi_9
\end{aligned} \tag{53}
$$

where

$$\phi_0 = 1 \tag{54}$$
$$\phi_1 = \frac{x}{\Delta x} \tag{55}$$
$$\phi_2 = \frac{y}{\Delta y} \tag{56}$$
$$\phi_3 = \frac{z}{\Delta z} \tag{57}$$
$$\phi_4 = \phi_1^2(x) - \frac{1}{12} \tag{58}$$
$$\phi_5 = \phi_1(x)\phi_2(y) \tag{59}$$
$$\phi_6 = \phi_2^2(y) - \frac{1}{12} \tag{60}$$
$$\phi_7 = \phi_2(y)\phi_3(z) \tag{61}$$
$$\phi_8 = \phi_3^2 - \frac{1}{12} \tag{62}$$
$$\phi_9 = \phi_1\phi_3 \tag{63}$$

and the variables to be evolved are now

$$u_0(t),\; u_x(t),\; u_y(t),\; u_z(t),\; u_{xx}(t),\; u_{xy}(t),\; u_{yy}(t),\; u_{yz}(t),\; u_{zz}(t),\; u_{zx}(t), \tag{64}$$

Note that the basis functions are still orthogonal, and therefore we have

$$\int_V \phi_i \phi_j \, dV = \Delta V \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{180} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{144} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{180} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{144} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{180} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{144} \end{bmatrix}. \tag{65}$$

The gradient of each basis is given by

$$\mathrm{grad}\,\phi_0 = 0 \tag{66}$$
$$\mathrm{grad}\,\phi_1 = \left(\frac{1}{\Delta x}, 0, 0\right) \tag{67}$$
$$\mathrm{grad}\,\phi_2 = \left(0, \frac{1}{\Delta y}, 0\right) \tag{68}$$
$$\mathrm{grad}\,\phi_3 = \left(0, 0, \frac{1}{\Delta z}\right). \tag{69}$$
$$\mathrm{grad}\,\phi_4 = \left(\frac{2\,\phi_1(x)}{\Delta x}, 0, 0\right) \tag{70}$$
$$\mathrm{grad}\,\phi_5 = \left(\frac{\phi_2(y)}{\Delta x}, \frac{\phi_1(x)}{\Delta y}, 0\right) \tag{71}$$
$$\mathrm{grad}\,\phi_6 = \left(0, \frac{2\,\phi_2(y)}{\Delta y}, 0\right). \tag{72}$$
$$\mathrm{grad}\,\phi_7 = \left(0, \frac{\phi_3(z)}{\Delta y}, \frac{\phi_2(y)}{\Delta z}\right) \tag{73}$$
$$\mathrm{grad}\,\phi_8 = \left(0, 0, \frac{2\,\phi_3(z)}{\Delta z}\right) \tag{74}$$
$$\mathrm{grad}\,\phi_9 = \left(\frac{\phi_3(z)}{\Delta x}, 0, \frac{\phi_1(x)}{\Delta z}\right). \tag{75}$$
$$\tag{76}$$

Similarly, on the faces, we define a piecewise quadratic variation.

$$B^x\left(x = \frac{\Delta x}{2}, y, z, t\right) = B_0^x(t)\psi_0 + B_y^x(t)\psi_1 + B_z^x(t)\psi_2 + B_{yy}^x(t)\psi_3 + B_{yz}^x(t)\psi_4 + B_{zz}^x(t)\psi_5$$

where

$$\psi_0 = 1,\ \psi_1 = \frac{y}{\Delta y},\ \psi_2 = \frac{z}{\Delta z},\ \psi_3 = \psi_1^2(y) - \frac{1}{12},\ \psi_4 = \psi_1(y)\psi_2(z),\ \psi_5 = \psi_2(z) - \frac{1}{12}$$

and the variables to be evolved are

$$B_0^x(t),\ \ B_y^x(t),\ \ B_z^x(t)),\ \ B_{yy}^x(t),\ \ B_{yz}^x(t)),\ \ B_{zz}^x(t). \tag{77}$$

Note that the basis functions are again orthogonal, and therefore we have

$$\int_S \psi_i \psi_j \, dS = \Delta y \Delta z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 0 & \frac{1}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{180} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{144} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{180} \end{bmatrix}. \tag{78}$$

The gradient of each basis is given by

$$\begin{aligned}
\text{grad}\,\psi_0 &= 0 && (79)\\
\text{grad}\,\psi_1 &= \left(0, \frac{1}{\Delta y}, 0\right) && (80)\\
\text{grad}\,\psi_2 &= \left(0, 0, \frac{1}{\Delta z}\right) && (81)\\
\text{grad}\,\psi_3 &= \left(0, \frac{2\,\psi_1(y)}{\Delta y}, 0\right) && (82)\\
\text{grad}\,\psi_4 &= \left(0, \frac{\psi_2(z)}{\Delta y}, \frac{\psi_1(y)}{\Delta z}\right) && (83)\\
\text{grad}\,\psi_5 &= \left(0, 0, \frac{2\,\psi_2(z)}{\Delta z}\right) && (84)\\
& && (85)
\end{aligned}$$

The variations of y-component and z-component, ($B^y$ and $B^z$) are defined analogously (a matter of a simple cyclic premutation among $x, y, z$). For example, we have, at $y = \frac{\Delta y}{2}$

$$\begin{aligned}
\text{grad}\,\psi_0 &= 0 && (86)\\
\text{grad}\,\psi_1 &= \left(0, 0, \frac{1}{\Delta z}\right) && (87)\\
\text{grad}\,\psi_2 &= \left(\frac{1}{\Delta x}, 0, 0\right) && (88)\\
\text{grad}\,\psi_3 &= \left(0, 0, \frac{2\,\psi_1(z)}{\Delta z}\right) && (89)\\
\text{grad}\,\psi_4 &= \left(\frac{\psi_1(z)}{\Delta x}, 0, \frac{\psi_2(x)}{\Delta z}\right) && (90)\\
\text{grad}\,\psi_5 &= \left(\frac{2\,\psi_2(x)}{\Delta x}, 0, 0\right) && (91)\\
& && (92)
\end{aligned}$$

and at $z = \frac{\Delta z}{2}$

$$\begin{aligned}
\text{grad}\,\psi_0 &= 0 && (93)\\
\text{grad}\,\psi_1 &= \left(\frac{1}{\Delta x}, 0, 0\right) && (94)\\
\text{grad}\,\psi_2 &= \left(0, \frac{1}{\Delta y}, 0\right) && (95)\\
\text{grad}\,\psi_3 &= \left(\frac{2\,\psi_1(x)}{\Delta x}, 0, 0\right) && (96)
\end{aligned}$$

$$\text{grad}\,\psi_4 = \left(\frac{\psi_2(y)}{\Delta x}, \frac{\psi_1(x)}{\Delta y}, 0\right) \tag{97}$$

$$\text{grad}\,\psi_5 = \left(0, \frac{2\,\psi_2(y)}{\Delta y}, 0\right). \tag{98}$$

$$\tag{99}$$

## 3.2 The Conservation Law

Because of orthogonality of the basis functions, the equations for the variables up to first-order remain intact. For the second-order moments, from

$$\int_V \partial_t \mathbf{U}_h\, \phi_j\, dV + \oint_{\partial V} \phi_j \mathbf{F}_n\, dS - \int_V \mathbf{F}\, \text{grad}\phi_j\, dV = 0. \tag{100}$$

we obtain

$$\frac{d\mathbf{U}_{xx}}{dt} = -\frac{180}{\Delta V}\left(\oint_{\partial V} \mathbf{F}_n\, \phi_4(x)\, dS - \frac{2}{\Delta x}\int_V F\, \phi_1(x)\, dV\right) \tag{101}$$

$$\frac{d\mathbf{U}_{xy}}{dt} = -\frac{144}{\Delta V}\left(\oint_{\partial V} \mathbf{F}_n\, \phi_5(x,y)\, dS - \frac{1}{\Delta x}\int_V F\, \phi_2(y)\, dV - \frac{1}{\Delta y}\int_V G\, \phi_1(x)\, dV\right) \tag{102}$$

$$\frac{d\mathbf{U}_{yy}}{dt} = -\frac{180}{\Delta V}\left(\oint_{\partial V} \mathbf{F}_n\, \phi_6(y)\, dS - \frac{2}{\Delta y}\int_V G\, \phi_2(y)\, dV\right) \tag{103}$$

$$\frac{d\mathbf{U}_{yz}}{dt} = -\frac{144}{\Delta V}\left(\oint_{\partial V} \mathbf{F}_n\, \phi_7(y,z)\, dS - \frac{1}{\Delta y}\int_V G\, \phi_3(z)\, dV - \frac{1}{\Delta z}\int_V H\, \phi_2(y)\, dV\right) \tag{104}$$

$$\frac{d\mathbf{U}_{zz}}{dt} = -\frac{180}{\Delta V}\left(\oint_{\partial V} \mathbf{F}_n\, \phi_8(z)\, dS - \frac{2}{\Delta z}\int_V H\, \phi_3(z)\, dV\right) \tag{105}$$

$$\frac{d\mathbf{U}_{zx}}{dt} = -\frac{144}{\Delta V}\left(\oint_{\partial V} \mathbf{F}_n\, \phi_9(z,x) dS - \frac{1}{\Delta x}\int_V F\, \phi_3(z)\, dV - \frac{1}{\Delta z}\int_V H\, \phi_1(x)\, dV\right) \tag{106}$$

Made more specific to a Cartesian cell, these equations are written

$$\begin{aligned}
\frac{1}{180}\frac{d\mathbf{U}_{xx}}{dt} &+ \frac{1}{\Delta x}\left[\int F\left(x=\frac{\Delta x}{2}, y, z\right)\left(\frac{1}{6}\right)\, dydz - \int F\left(x=-\frac{\Delta x}{2}, y, z\right)\left(\frac{1}{6}\right)\, dydz\right]\frac{1}{\Delta y \Delta z} \\
&+ \frac{1}{\Delta y}\left[\int G\left(x, y=\frac{\Delta y}{2}, z\right)\phi_4(x)\, dxdz - \int G\left(x, y=-\frac{\Delta y}{2}, z\right)\phi_4(x)\, dxdz\right]\frac{1}{\Delta x \Delta z} \\
&+ \frac{1}{\Delta z}\left[\int H\left(x, y, z=\frac{\Delta z}{2}\right)\phi_4(x)\, dxdy - \int H\left(x, y, z=-\frac{\Delta z}{2}\right)\phi_4(x)\, dxdy\right]\frac{1}{\Delta x \Delta y} \\
&- \frac{2}{\Delta x}\left[\int_V F\left(x,y,z\right)\,\phi_4(x)\, dxdydz\right]\frac{1}{\Delta x \Delta y \Delta z} = 0
\end{aligned} \tag{107}$$

$$\begin{aligned}
\frac{1}{144}\frac{d\mathbf{U}_{xy}}{dt} &+ \frac{1}{\Delta x}\left[\int F\left(x=\frac{\Delta x}{2}, y, z\right)\left(\frac{1}{2}\right)\,\phi_2(y)\, dydz - \int F\left(x=-\frac{\Delta x}{2}, y, z\right)\left(-\frac{1}{2}\right)\,\phi_2(y)\, dydz\right]\frac{1}{\Delta y \Delta z} \\
&+ \frac{1}{\Delta y}\left[\int G\left(x, y=\frac{\Delta y}{2}, z\right)\phi_1(x)\left(\frac{1}{2}\right)\, dxdz - \int G\left(x, y=-\frac{\Delta y}{2}, z\right)\phi_1(x)\left(-\frac{1}{2}\right)\, dxdz\right]\frac{1}{\Delta x \Delta z} \\
&+ \frac{1}{\Delta z}\left[\int H\left(x, y, z=\frac{\Delta z}{2}\right)\phi_5(x,y)\, dxdy - \int H\left(x, y, z=-\frac{\Delta z}{2}\right)\phi_5(x,y)\, dxdy\right]\frac{1}{\Delta x \Delta y} \\
&- \frac{1}{\Delta x}\left[\int_V F\left(x,y,z\right)\,\phi_2(y)\, dxdydz\right]\frac{1}{\Delta x \Delta y \Delta z} - \frac{1}{\Delta y}\left[\int_V G\left(x,y,z\right)\,\phi_1(x)\, dxdydz\right]\frac{1}{\Delta x \Delta y \Delta z} = 0
\end{aligned} \tag{108}$$

$$
\begin{aligned}
\frac{1}{180}\frac{d\mathbf{U}_{yy}}{dt} &+ \frac{1}{\Delta x}\left[\int F\left(x=\frac{\Delta x}{2},y,z\right)\phi_6(y)\,dydz-\int F\left(x=-\frac{\Delta x}{2},y,z\right)\phi_6(y)\,dydz\right]\frac{1}{\Delta y\Delta z}\\
&+ \frac{1}{\Delta y}\left[\int G\left(x,y=\frac{\Delta y}{2},z\right)\left(\frac{1}{6}\right)\,dxdz-\int G\left(x,y=-\frac{\Delta y}{2},z\right)\left(\frac{1}{6}\right)\,dxdz\right]\frac{1}{\Delta x\Delta z}\\
&+ \frac{1}{\Delta z}\left[\int H\left(x,y,z=\frac{\Delta z}{2}\right)\phi_6(y)\,dxdy-\int H\left(x,y,z=-\frac{\Delta z}{2}\right)\phi_6(y)\,dxdy\right]\frac{1}{\Delta x\Delta y}\\
&- \frac{2}{\Delta y}\left[\int_V G\left(x,y,z\right)\,\phi_2(y)\,dxdydz\right]\frac{1}{\Delta x\Delta y\Delta z}=0
\end{aligned}
\tag{109}
$$

$$
\begin{aligned}
\frac{1}{144}\frac{d\mathbf{U}_{yz}}{dt} &+ \frac{1}{\Delta x}\left[\int F\left(x=\frac{\Delta x}{2},y,z\right)\phi_7(y,z)\,dydz-\int F\left(x=-\frac{\Delta x}{2},y,z\right)\phi_7(y,z)\,dydz\right]\frac{1}{\Delta y\Delta z}\\
&+ \frac{1}{\Delta y}\left[\int G\left(x,y=\frac{\Delta y}{2},z\right)\phi_3(z)\left(\frac{1}{2}\right)\,dxdz-\int G\left(x,y=-\frac{\Delta y}{2},z\right)\phi_3(z)\left(-\frac{1}{2}\right)\,dxdz\right]\frac{1}{\Delta x\Delta z}\\
&+ \frac{1}{\Delta z}\left[\int H\left(x,y,z=\frac{\Delta z}{2}\right)\phi_2(y)\left(\frac{1}{2}\right)\,dxdy-\int H\left(x,y,z=-\frac{\Delta z}{2}\right)\phi_2(y)\left(-\frac{1}{2}\right)\,dxdy\right]\frac{1}{\Delta x\Delta y}\\
&- \frac{1}{\Delta y}\left[\int_V G\left(x,y,z\right)\,\phi_3(z)\,dxdydz\right]\frac{1}{\Delta x\Delta y\Delta z}-\frac{1}{\Delta z}\left[\int_V H\left(x,y,z\right)\,\phi_2(y)\,dxdydz\right]\frac{1}{\Delta x\Delta y\Delta z}=0
\end{aligned}
\tag{110}
$$

$$
\begin{aligned}
\frac{1}{180}\frac{d\mathbf{U}_{zz}}{dt} &+ \frac{1}{\Delta x}\left[\int F\left(x=\frac{\Delta x}{2},y,z\right)\phi_8(y)\,dydz-\int F\left(x=-\frac{\Delta x}{2},y,z\right)\phi_8(y)\,dydz\right]\frac{1}{\Delta y\Delta z}\\
&+ \frac{1}{\Delta y}\left[\int G\left(x,y=\frac{\Delta y}{2},z\right)\phi_8(y)\,dxdz-\int G\left(x,y=-\frac{\Delta y}{2},z\right)\phi_8(y)\,dxdz\right]\frac{1}{\Delta x\Delta z}\\
&+ \frac{1}{\Delta z}\left[\int H\left(x,y,z=\frac{\Delta z}{2}\right)\left(\frac{1}{6}\right)dxdy-\int H\left(x,y,z=-\frac{\Delta z}{2}\right)\left(\frac{1}{6}\right)dxdy\right]\frac{1}{\Delta x\Delta y}\\
&- \frac{2}{\Delta z}\left[\int_V H\left(x,y,z\right)\,\phi_3(z)\,dxdydz\right]\frac{1}{\Delta x\Delta y\Delta z}=0
\end{aligned}
\tag{111}
$$

### 3.3 The Stokes' Law

In the same way, from

$$
\int_V \partial_t\mathbf{U}_h\,\phi_j\,dV+\oint_{\partial S}\phi_j\mathbf{E}\cdot d\ell-\int_S(\operatorname{grad}\phi_j\times\mathbf{E})\cdot d\mathbf{S}=0. \tag{112}
$$

we obtain over the face at $x=\frac{\Delta x}{2}$

$$
\begin{aligned}
\frac{dB^x_{yy}}{dt} &= -\frac{180}{\Delta S}\left(\oint_{\partial S}\psi_3(y)\mathbf{E}\cdot d\ell-\frac{2}{\Delta y}\int_S E_z\,\psi_1(y)\,dS\right)\\
\frac{dB^x_{yz}}{dt} &= -\frac{144}{\Delta S}\left(\oint_{\partial S}\psi_4(z)\mathbf{E}\cdot d\ell-\frac{1}{\Delta y}\int_S E_z\,\psi_2(z)\,dS+\frac{1}{\Delta z}\int_S E_y\,\psi_1(y)\,dS\right)\\
\frac{dB^x_{zz}}{dt} &= -\frac{180}{\Delta S}\left(\oint_{\partial S}\psi_5(z)\mathbf{E}\cdot d\ell+\frac{2}{\Delta z}\int_S E_y\,\psi_2(z)\,dS\right)
\end{aligned}
$$

Similarly, at $y=\frac{\Delta y}{2}$

$$
\begin{aligned}
\frac{dB^y_{zz}}{dt} &= -\frac{180}{\Delta S}\left(\oint_{\partial S}\psi_3(z)\mathbf{E}\cdot d\ell-\frac{2}{\Delta z}\int_S E_x\,\psi_1(z)\,dS\right)\\
\frac{dB^y_{zx}}{dt} &= -\frac{144}{\Delta S}\left(\oint_{\partial S}\psi_4(x)\mathbf{E}\cdot d\ell-\frac{1}{\Delta z}\int_S E_x\,\psi_2(x)\,dS+\frac{1}{\Delta x}\int_S E_z\,\psi_1(z)\,dS\right)\\
\frac{dB^y_{xx}}{dt} &= -\frac{180}{\Delta S}\left(\oint_{\partial S}\psi_5(x)\mathbf{E}\cdot d\ell+\frac{2}{\Delta x}\int_S E_z\,\psi_2(x)\,dS\right)
\end{aligned}
$$

and at $z = \frac{\Delta z}{2}$

$$
\begin{aligned}
\frac{dB^z_{xx}}{dt} &= -\frac{180}{\Delta S}\left(\oint_{\partial S} \psi_3(x)\mathbf{E}\cdot d\ell - \frac{2}{\Delta x}\int_S E_y\,\psi_1(x)\,dS\right) \\
\frac{dB^z_{xy}}{dt} &= -\frac{144}{\Delta S}\left(\oint_{\partial S} \psi_4(y)\mathbf{E}\cdot d\ell - \frac{1}{\Delta x}\int_S E_y\,\psi_2(y)\,dS + \frac{1}{\Delta y}\int_S E_x\,\psi_1(x)\,dS\right) \\
\frac{dB^z_{yy}}{dt} &= -\frac{180}{\Delta S}\left(\oint_{\partial S} \psi_5(y)\mathbf{E}\cdot d\ell + \frac{2}{\Delta y}\int_S E_x\,\psi_2(y)\,dS\right)
\end{aligned}
$$

Made more specific to a Cartesian cell, these equations are written, at $x = \frac{\Delta x}{2}$,

$$
\begin{aligned}
\frac{1}{180}\frac{dB^x_{yy}}{dt} &+ \frac{1}{\Delta y\Delta z}\left\{\int E_z\left(y=\frac{\Delta y}{2},z\right)\left(\frac{1}{6}\right)dz - \int E_z\left(y=-\frac{\Delta y}{2},z\right)\left(\frac{1}{6}\right)dz\right. \\
&\left. - \int E_y\left(y,z=\frac{\Delta z}{2}\right)\psi_3(y)\,dy + \int E_y\left(y,z=-\frac{\Delta z}{2}\right)\psi_3(y)\,dy\right\} \\
&- \frac{2}{\Delta y}\left[\int_S E_z\,\psi_1(y)\,dydz\right]\frac{1}{\Delta y\Delta z} = 0
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
\frac{1}{144}\frac{dB^x_{yz}}{dt} &+ \frac{1}{\Delta y\Delta z}\left\{\int E_z\left(y=\frac{\Delta y}{2},z\right)\left(\frac{1}{2}\right)\psi_2(z)\,dz - \int E_z\left(y=-\frac{\Delta y}{2},z\right)\left(-\frac{1}{2}\right)\psi_2(z)\,dz\right. \\
&\left. - \int E_y\left(y,z=\frac{\Delta z}{2}\right)\left(\frac{1}{2}\right)\psi_1(y)\,dy + \int E_y\left(y,z=-\frac{\Delta z}{2}\right)\left(-\frac{1}{2}\right)\psi_1(y)\,dy\right\} \\
&- \frac{1}{\Delta y}\left[\int_S E_z\,\psi_2(z)\,dydz\right]\frac{1}{\Delta y\Delta z} + \frac{1}{\Delta z}\left[\int_S E_y\,\psi_1(y)\,dydz\right]\frac{1}{\Delta y\Delta z} = 0
\end{aligned}
\tag{114}
$$

$$
\begin{aligned}
\frac{1}{180}\frac{dB^x_{zz}}{dt} &+ \frac{1}{\Delta y\Delta z}\left\{\int E_z\left(y=\frac{\Delta y}{2},z\right)\psi_5(z)\,dz - \int E_z\left(y=-\frac{\Delta y}{2},z\right)\psi_5(z)\,dz\right. \\
&\left. - \int E_y\left(y,z=\frac{\Delta z}{2}\right)\left(\frac{1}{6}\right)dy + \int E_y\left(y,z=-\frac{\Delta z}{2}\right)\left(\frac{1}{6}\right)dy\right\} \\
&+ \frac{2}{\Delta z}\left[\int_S E_y\,\psi_2(z)\,dydz\right]\frac{1}{\Delta y\Delta z} = 0
\end{aligned}
\tag{115}
$$

Similarly, at $y = \frac{\Delta y}{2}$,

$$
\begin{aligned}
\frac{1}{180}\frac{dB^y_{zz}}{dt} &+ \frac{1}{\Delta z\Delta x}\left\{\int E_x\left(z=\frac{\Delta z}{2},x\right)\left(\frac{1}{6}\right)dx - \int E_x\left(z=-\frac{\Delta z}{2},x\right)\left(\frac{1}{6}\right)dx\right. \\
&\left. - \int E_z\left(z,x=\frac{\Delta x}{2}\right)\psi_3(z)\,dz + \int E_z\left(z,x=-\frac{\Delta x}{2}\right)\psi_3(z)\,dz\right\} \\
&- \frac{2}{\Delta z}\left[\int_S E_x\,\psi_1(z)\,dzdx\right]\frac{1}{\Delta z\Delta x} = 0
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
\frac{1}{144}\frac{dB^y_{zx}}{dt} &+ \frac{1}{\Delta z\Delta x}\left\{\int E_x\left(z=\frac{\Delta z}{2},x\right)\left(\frac{1}{2}\right)\psi_2(x)\,dx - \int E_x\left(z=-\frac{\Delta z}{2},x\right)\left(-\frac{1}{2}\right)\psi_2(x)\,dx\right. \\
&\left. - \int E_z\left(z,x=\frac{\Delta x}{2}\right)\left(\frac{1}{2}\right)\psi_1(z)\,dz + \int E_z\left(z,x=-\frac{\Delta x}{2}\right)\left(-\frac{1}{2}\right)\psi_1(z)\,dz\right\} \\
&- \frac{1}{\Delta z}\left[\int_S E_x\,\psi_2(x)\,dzdx\right]\frac{1}{\Delta z\Delta x} + \frac{1}{\Delta x}\left[\int_S E_z\,\psi_1(z)\,dzdx\right]\frac{1}{\Delta z\Delta x} = 0
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
\frac{1}{180}\frac{dB^y_{xx}}{dt} &+ \frac{1}{\Delta z \Delta x}\left\{\int E_x\left(z=\frac{\Delta z}{2},x\right)\psi_5(x)\,dx - \int E_x\left(z=-\frac{\Delta z}{2},x\right)\psi_5(x)\,dx \right.\\
&\left. - \int E_z\left(z,x=\frac{\Delta x}{2}\right)\left(\frac{1}{6}\right)dz + \int E_z\left(z,x=-\frac{\Delta x}{2}\right)\left(\frac{1}{6}\right)dz\right\}\\
&+ \frac{2}{\Delta x}\left[\int_S E_z\,\psi_2(x)\,dzdx\right]\frac{1}{\Delta z\Delta x} = 0
\end{aligned} \tag{118}
$$

Similarly, at $z = \frac{\Delta z}{2}$,

$$
\begin{aligned}
\frac{1}{180}\frac{dB^z_{xx}}{dt} &+ \frac{1}{\Delta x \Delta y}\left\{\int E_y\left(x=\frac{\Delta x}{2},y\right)\left(\frac{1}{6}\right)dy - \int E_y\left(x=-\frac{\Delta x}{2},y\right)\left(\frac{1}{6}\right)dy \right.\\
&\left. - \int E_x\left(x,y=\frac{\Delta y}{2}\right)\psi_3(x)\,dx + \int E_x\left(x,y=-\frac{\Delta y}{2}\right)\psi_3(x)\,dx\right\}\\
&- \frac{2}{\Delta x}\left[\int_S E_y\,\psi_1(x)\,dxdy\right]\frac{1}{\Delta x\Delta y} = 0
\end{aligned} \tag{119}
$$

$$
\begin{aligned}
\frac{1}{144}\frac{dB^z_{xy}}{dt} &+ \frac{1}{\Delta x \Delta y}\left\{\int E_y\left(x=\frac{\Delta x}{2},y\right)\left(\frac{1}{2}\right)\psi_2(y)\,dy - \int E_y\left(x=-\frac{\Delta x}{2},y\right)\left(-\frac{1}{2}\right)\psi_2(y)\,dy \right.\\
&\left. - \int E_x\left(x,y=\frac{\Delta y}{2}\right)\left(\frac{1}{2}\right)\psi_1(x)\,dx + \int E_x\left(x,y=-\frac{\Delta y}{2}\right)\left(-\frac{1}{2}\right)\psi_1(x)\,dx\right\}\\
&- \frac{1}{\Delta x}\left[\int_S E_y\,\psi_2(y)\,dxdy\right]\frac{1}{\Delta x\Delta y} + \frac{1}{\Delta y}\left[\int_S E_x\,\psi_1(x)\,dxdy\right]\frac{1}{\Delta x\Delta y} = 0
\end{aligned} \tag{120}
$$

$$
\begin{aligned}
\frac{1}{180}\frac{dB^z_{yy}}{dt} &+ \frac{1}{\Delta x \Delta y}\left\{\int E_y\left(x=\frac{\Delta x}{2},y\right)\psi_5(y)\,dy - \int E_y\left(x=-\frac{\Delta x}{2},y\right)\psi_5(y)\,dy \right.\\
&\left. - \int E_x\left(x,y=\frac{\Delta y}{2}\right)\left(\frac{1}{6}\right)dx + \int E_x\left(x,y=-\frac{\Delta y}{2}\right)\left(\frac{1}{6}\right)dx\right\}\\
&+ \frac{2}{\Delta y}\left[\int_S E_x\,\psi_2(y)\,dxdy\right]\frac{1}{\Delta x\Delta y} = 0
\end{aligned} \tag{121}
$$

# 4 Time Integration

To integrate the equations derived in the previous sections, we employ the second-order and third-order accurate Runge-Kutta schemes as in Shu's paper for $P^1$ and $P^2$ discretization respectively. Writing the equations derived above in the form

$$
\frac{\partial \mathbf{U}}{\partial t} = L(\mathbf{U}), \tag{122}
$$

we implement the second-order scheme in the form

$$
\mathbf{U}^{(1)} = \mathbf{U}^n + \Delta t\, L(\mathbf{U}^n) \tag{123}
$$

$$
\mathbf{U}^{n+1} = \frac{1}{2}\left(\mathbf{U}^n + \mathbf{U}^{(1)}\right) + \frac{1}{2}\Delta t\, L(\mathbf{U}^{(1)}), \tag{124}
$$

and the third-order scheme in the form

$$
\mathbf{U}^{(1)} = \mathbf{U}^n + \Delta t L(\mathbf{U}^n) \tag{125}
$$

$$
\mathbf{U}^{(2)} = \frac{1}{4}\left(3\,\mathbf{U}^n + \mathbf{U}^{(1)}\right) + \frac{1}{4}\Delta t\, L(\mathbf{U}^{(1)}) \tag{126}
$$

$$
\mathbf{U}^{n+1} = \frac{1}{3}\left(\mathbf{U}^n + 2\,\mathbf{U}^{(2)}\right) + \frac{2}{3}\Delta t\, L(\mathbf{U}^{(2)}). \tag{127}
$$