Roe Flux in Deforming Mesh

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1 Governing Equation

Consider the Euler equations over a deforming control volume:

$$\frac{\partial}{\partial t}\int_V \mathbf{U}dV + \oint_{\partial V}(\mathbf{F}(\mathbf{U}) - \mathbf{U}\otimes\mathbf{w})\cdot\mathbf{n}dS = 0, \tag{1}$$

where $\mathbf{U} = [\rho, \rho u, \rho v, \rho w, \rho E(\mathbf{u})]^t$ is a vector of conservative variables, $\mathbf{F}$ is a convective flux tensor whose normal projection is given by

$$\mathbf{F}_n(\mathbf{U}) = \mathbf{F}(\mathbf{U})\cdot\mathbf{n} = [\rho u_n, \rho u_n u + p n_x, \rho u_n v + p n_y, \rho u_n w + p n_z, \rho u_n H]^t, \tag{2}$$

$\mathbf{w} = [w_x, w_y, w_z]^t$ is a local control volume face velocity, and

$$E(\mathbf{u}) = \frac{p}{\gamma-1} + \frac{\rho\mathbf{u}^2}{2} = \frac{p}{\gamma-1} + \frac{\rho}{2}(u^2+v^2+w^2). \tag{3}$$

Following Ref.[1], we introduce a matrix $\mathbf{T}$ defined by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -w_x & 1 & 0 & 0 & 0 \\ -w_y & 0 & 1 & 0 & 0 \\ -w_z & 0 & 0 & 1 & 0 \\ \mathbf{w}^2/2 & -w_x & -w_y & -w_z & 1 \end{bmatrix}, \quad \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ w_x & 1 & 0 & 0 & 0 \\ w_y & 0 & 1 & 0 & 0 \\ w_z & 0 & 0 & 1 & 0 \\ \mathbf{w}^2/2 & w_x & w_y & w_z & 1 \end{bmatrix}. \tag{4}$$

Then, $\mathbf{TU}$ gives a vector of conservative variables based on the relative velocity:

$$\mathbf{TU} = [\rho, \rho(u-w_x), \rho(v-w_y), \rho(w-w_z), \rho E(\mathbf{u}-\mathbf{w})]^t, \tag{5}$$

and we have

$$(\mathbf{F}(\mathbf{U}) - \mathbf{U}\otimes\mathbf{w})\cdot\mathbf{n} = \mathbf{T}^{-1}\mathbf{F}_n(\mathbf{TU}). \tag{6}$$

and so we can write Equation (1) as

$$\frac{\partial}{\partial t}\int_V \mathbf{U}dV + \oint_{\partial V}\mathbf{T}^{-1}\mathbf{F}_n(\mathbf{TU})dS = 0, \tag{7}$$

A finite-volume discretization yields the following semi-discrete scheme:

$$\frac{d(V_j\mathbf{U}_j)}{dt} = \mathbf{Res}_j, \tag{8}$$

where $V_j\mathbf{U}_j$ is a vector of the product of the numerical solution and the control volume at a node j, $\mathbf{Res_j}$ is the residual of the spatial operator at j. The residual is given by

$$\mathbf{Res}_j = \sum_{k\in\{k_j\}}\Phi_{ij}\left(\mathbf{U}_L, \mathbf{U}_R\right)A_{jk}, \tag{9}$$

where A_{jk} denotes the face area, and Φ_{ij} is a numerical flux. The numerical flux can be constructed a product of $\mathbf{T}^{-1}$ and a numerical flux for $\mathbf{F}_n(\mathbf{TU})$, i.e., a numerical flux based on the relative velocity. For example, the Roe flux can be constructed as

$$\Phi_{ij}\left(\mathbf{U}_L, \mathbf{U}_R\right) = \mathbf{T}^{-1}\left[\frac{1}{2}(\mathbf{F}_n(\mathbf{TU}_L) + \mathbf{F}_n(\mathbf{TU}_R) - \left|\mathbf{A}_n(\mathbf{T}\hat{\mathbf{U}})\right|(\mathbf{TU}_R - \mathbf{TU}_L))\right], \tag{10}$$

where the hat denotes the Roe average, and

$$\mathbf{A}_n(\mathbf{T}\hat{\mathbf{U}}) = \frac{\partial \mathbf{F}_n(\mathbf{T}\hat{\mathbf{U}})}{\partial(\mathbf{T}\hat{\mathbf{U}})}, \quad |\mathbf{A}_n| = \mathbf{R}\,|\Lambda_n|\,\mathbf{R}^{-1}. \tag{11}$$

$$\mathbf{\Lambda} = \begin{bmatrix} (u_n - w_n) - c & 0 & 0 & 0 & 0 \\ 0 & (u_n - w_n) & 0 & 0 & 0 \\ 0 & 0 & (u_n - w_n) + c & 0 & 0 \\ 0 & 0 & 0 & (u_n - w_n) & 0 \\ 0 & 0 & 0 & 0 & (u_n - w_n) \end{bmatrix}, \tag{12}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ (u - w_x) - c\,n_x & (u - w_x) & (u - w_x) + c\,n_x & \ell_x & m_x \\ (v - w_y) - c\,n_y & (v - w_y) & (v - w_y) + c\,n_y & \ell_y & m_y \\ (w - w_z) - c\,n_z & (w - w_z) & (w - w_z) + c\,n_z & \ell_z & m_z \\ H - (u_n - w_n)c & (\mathbf{u} - \mathbf{w})^2/2 & H + (u_n - w_n)c & (u_\ell - w_\ell) & (u_m - w_m) \end{bmatrix}, \tag{13}$$

$$\mathbf{R}^{-1}\Delta\mathbf{U} = \left[\frac{\Delta p - \rho c\,\Delta u_n}{2c^2}, -\frac{\Delta p - c^2\,\Delta\rho}{c^2}, \frac{\Delta p + \rho c\,\Delta u_n}{2c^2}, \rho\,\Delta u_\ell, \rho\,\Delta u_m\right]^t. \tag{14}$$

$$u_n = (u, v, w)\cdot(n_x, n_y, n_z), \quad u_\ell = (u, v, w)\cdot(\ell_x, \ell_y, \ell_z), \quad u_m = (u, v, w)\cdot(m_x, m_y, m_z). \tag{15}$$

Note that Δu_n is independent of $\mathbf{w}$, and similarly for other velocity differences. The ambiguous face-tangent vectors, (ℓ_x, ℓ_y, ℓ_z) and (m_x, m_y, m_z) can be totally eliminated as described in Ref.[2].

The Roe flux can be implemented in the pre-multiplied form:

$$\Phi_{ij}\left(\mathbf{U}_L, \mathbf{U}_R\right) = \frac{1}{2}(\mathbf{F}_n(\mathbf{U}_L) - \mathbf{U}_L\mathbf{w}_n + \mathbf{F}_n(\mathbf{U}_R) - \mathbf{U}_R\mathbf{w}_n) - \frac{1}{2}\mathbf{R}^a\,|\Lambda_n|\,\mathbf{R}^{-1}\Delta\mathbf{U}, \tag{16}$$

where

$$\mathbf{R}^a = \mathbf{T}^{-1}\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ u - c\,n_x & u & u + c\,n_x & \ell_x & m_x \\ v - c\,n_y & v & v + c\,n_y & \ell_y & m_y \\ w - c\,n_z & w & w + c\,n_z & \ell_z & m_z \\ H - u_n c & \mathbf{u}^2/2 & H + u_n c & u_\ell & u_m \end{bmatrix}. \tag{17}$$

Therefore, in the Roe dissipation matrix, the right eigenvectors should be evaluated by the absolute velocity, not by the relative velocity. The relative velocity is used only in $|\Lambda_n|$. It does not matter for $\mathbf{R}^{-1}\Delta\mathbf{U}$ as explained earlier.

References

[1] D. Muffo, G. Quaranta, A. Guardone, and P. Mantegazza. Interface velocity consistency in time-accurate flow simulations on dynamic meshes. Scientific Report DIA-SR 07-01, Politecnico di Milano, 2007.

[2] Katate Masatsuka. I do like CFD, VOL.1, Second Edition, version 2.4. `http://www.cfdbooks.com`, 2017.