Three-Dimensional Rotated-Hybrid Riemann Solvers

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1 Three-Dimensional Rotate-Hybrid Solvers

In three dimensions, the rotated Riemann solvers [1] require, in general, the decomposition of the normal vector $\mathbf{n}$ into three linearly independent directions. Let $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ be the unit vectors in the three directions. For simplicity, we assume that these directions are orthogonal:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0, \quad \mathbf{n}_2 \cdot \mathbf{n}_3 = 0, \quad \mathbf{n}_3 \cdot \mathbf{n}_1 = 0. \tag{1.1}$$

Note that we can select only two of the three vectors, say $\mathbf{n}_1$ and $\mathbf{n}_2$; the remaining vector, $\mathbf{n}_3$, is uniquely determined by orthogonality and by the direction of the face normal as described below. The cell-face normal $\mathbf{n}$ is then written as

$$\mathbf{n} = \alpha_1 \, \mathbf{n}_1 + \alpha_2 \, \mathbf{n}_2 + \alpha_3 \, \mathbf{n}_3, \tag{1.2}$$

where

$$\alpha_1 = \mathbf{n} \cdot \mathbf{n}_1, \quad \alpha_1 = \mathbf{n} \cdot \mathbf{n}_2, \quad \alpha_1 = \mathbf{n} \cdot \mathbf{n}_3. \tag{1.3}$$

Here, we choose these vectors such that $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $\alpha_3 \geq 0$, so that we keep the same left and right states in all directions. The interface flux is then decomposed as

$$\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{n}) = \alpha_1 \, \mathbf{\Phi}(\mathbf{n}_1) + \alpha_2 \, \mathbf{\Phi}(\mathbf{n}_2) + \alpha_3 \, \mathbf{\Phi}(\mathbf{n}_3). \tag{1.4}$$

1.1 Rotated-RHLL-Rusanov

If we choose the HLL flux in $\mathbf{n}_1$, and the Roe flux in $\mathbf{n}_2$ and the Rusanov flux in $\mathbf{n}_3$, then the rotated-flux (1.4) becomes

$$\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{n}) = \frac{S_R^+ \mathbf{H}_n(\mathbf{U}_L) - S_L^- \mathbf{H}_n(\mathbf{U}_R)}{S_R^+ - S_L^-} - \frac{1}{2} \sum_{k=1}^{4} |\hat{s}^k| \, \hat{w}_{n_2}^k \hat{\mathbf{r}}_{n_2}^k, \tag{1.5}$$

where

$$|\hat{s}^k| \quad = \quad \alpha_2 |\hat{\lambda}_{n_2}^k|^* - \frac{1}{S_R^+ - S_L^-} \left[\alpha_2 (S_R^+ + S_L^-) \hat{\lambda}_{n_2}^k + 2\alpha_1 S_R^+ S_L^- \right] + \alpha_3 \hat{S}_{n_3}, \tag{1.6}$$

where S_R^+ and S_L^- are computed based on $\mathbf{n}_2$, and $\hat{S}_{n_3}$ is the maximum wave speed in $\mathbf{n}_3$. This flux requires the eigenvectors only in $\mathbf{n}_2$; the other two normal vectors are needed only to compute the wave speed. Hence, it can be implemented as simply as a single Roe-type flux. It is very dissipative in the third direction, $\mathbf{n}_3$, but its impact may turn out to be negligiblly small. If the third direction happens to be perpendicular to the geometric face normal (i.e., if $\alpha_3 = 0$), $\mathbf{n}$, then this flux reduces to the Rotated-RHLL flux in Ref.[1].

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1.2 Rotated-RHLL-Roe

If we use the Roe flux (without an entropy fix) instead of the Rusanov flux in the third direction, we obtain the following roated-hybrid flux:

$$\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{n}) = \frac{S_R^+ \mathbf{H}_n(\mathbf{U}_L) - S_L^- \mathbf{H}_n(\mathbf{U}_R)}{S_R^+ - S_L^-} - \frac{1}{2} \sum_{k=1}^{4} \left(|\hat{s}_{RHLL}^k| \, \hat{w}_{n_2}^k \hat{\mathbf{r}}_{n_2}^k + 2 \hat{s}_3^k \hat{w}_{n_3}^k \hat{\mathbf{r}}_{n_3}^k \right), \tag{1.7}$$

where

$$|\hat{s}_{RHLL}^k| = \alpha_2 |\hat{\lambda}_{n_2}^k|^* - \frac{1}{S_R^+ - S_L^-} \left[\alpha_2 (S_R^+ + S_L^-) \hat{\lambda}_{n_2}^k + 2\alpha_1 S_R^+ S_L^- \right], \tag{1.8}$$

$$\hat{s}_3^k = \alpha_3 \frac{S_R^+ \left(\hat{\lambda}_{n_3}^k \right)^- + S_L^- \left(\hat{\lambda}_{n_3}^k \right)^+}{S_R^+ - S_L^-}. \tag{1.9}$$

This flux is more expensive than a single Roe-type flux since it requires two sets of eigenvectors: one with $\mathbf{n}_2$ and the other with $\mathbf{n}_3$.

2 Defining Directions

Following the previous work [1], we consider aligning $\mathbf{n}_1$ with the velocity-difference vector taken over two adjacent cells,

$$\mathbf{n}_1 = \hat{\Delta} \vec{q} \tag{2.1}$$

where

$$\hat{\Delta} \vec{q} = \frac{\Delta \vec{q}}{||\Delta \vec{q}||}, \tag{2.2}$$

$$\Delta \vec{q} = (\Delta u, \Delta v, \Delta w), \tag{2.3}$$

$$||\Delta \vec{q}|| = \sqrt{(\Delta u)^2 + (\Delta v)^2 + (\Delta w)^2}. \tag{2.4}$$

We assume for a moment that $||\Delta \vec{q}|| \neq 0$. For a second-order accurate scheme, the velocity difference is $O(h^3)$ for a smooth flow. But it is $O(1)$ if a discontinuity, such as a shock or shear wave, exists across the two cells; it then indicates the direction normal to a shock wave or parallel to a shear wave. For well-resolved flows, therefore, $\mathbf{n}_1$ may be considered as automatically aligned either normal to a shock wave or parallel to a shear wave. This means that the HLL flux is applied in these directions. For a shock wave, we expect it to suppress nonlinear instanbility. For a shear wave, it introduces a fair amount of dissiparion, but it depends on the relation of the geometric face normal and the shear wave (α_1 would be small if the wave is nearly parallel to the face).

The second direction, $\mathbf{n}_2$, corresponds to the Roe flux. It is required to be orthogonal to $\mathbf{n}_1$, thus lying in the plane normal to $\mathbf{n}_1$. One direction needs to be selected from infinitely many directions in the normal plane. Suppose that a shock wave is present across the interface. If it is a normal shock, the velocity in the normal plane is zero. Then, the dissipation of the second flux becomes independent of the direction, accounting only for a possible isotropic acoustic wave. Therefore, it may be chosen freely, or for some other consideration (e.g., shock instability or the magnitude of α_2). If the shock is an oblique shock, the velocity in the normal plane is finite but unchanged across the shock. We may then align $\mathbf{n}_2$ with this tangential velocity vector. In this case, the third direction will have no velocity components, and the dissipaion is introduced due to an isotropic acoustic wave only. On the other hand, if a shear wave is present across the interface, it would be reasonable to align $\mathbf{n}_2$ with the direction normal to (across) the shear wave to minimize the dissipation. This direction can be detected by a local density gradient, $\nabla \rho$, which should be available in a second-order Euler code. Note that the density gradient is not necessarily orthogonal to $\mathbf{n}_1$. Therefore, we select the direction in the normal plane closest to the density gradient. Once $\mathbf{n}_1$ and $\mathbf{n}_2$ are defined, the third direction, $\mathbf{n}_3$, is uniquely determined by orthogonality and the condition $\alpha_3 > 0$. Finally, if no waves are detected, i.e., $||\Delta \vec{q}|| = 0$ and $||\nabla \rho|| = 0$, then we may use the Roe flux in the geometric face normal direction: $\mathbf{n}_2 = \mathbf{n}$. In this case, we have $\alpha_1 = \alpha_3 = 0$ independently of the choice of the other two directions.

Taking all the above scenarios into account, we define the directions as follows:

1. **If shock wave**:

$$\mathbf{n}_1 = \hat{\Delta}\vec{q}, \tag{2.5}$$

$$\mathbf{n}_2 = \frac{\vec{q}_L - (\vec{q}_L \cdot \hat{\Delta}\vec{q})\hat{\Delta}\vec{q}}{||\vec{q}_L - (\vec{q}_L \cdot \hat{\Delta}\vec{q})\hat{\Delta}\vec{q}||}, \tag{2.6}$$

$$\mathbf{n}_3 = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{||\mathbf{n}_1 \times \mathbf{n}_2||}. \tag{2.7}$$

2. **If shear wave**:

$$\mathbf{n}_1 = \hat{\Delta}\vec{q}, \tag{2.8}$$

$$\mathbf{n}_2 = \frac{(\mathbf{n}_1 \times \nabla\rho) \times \mathbf{n}_1}{||(\mathbf{n}_1 \times \nabla\rho) \times \mathbf{n}_1||}, \tag{2.9}$$

$$\mathbf{n}_3 = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{||\mathbf{n}_1 \times \mathbf{n}_2||}. \tag{2.10}$$

3. **Otherwise**:

$$\mathbf{n}_1 = \text{any parallel direction}, \tag{2.11}$$

$$\mathbf{n}_2 = \mathbf{n}, \tag{2.12}$$

$$\mathbf{n}_3 = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{||\mathbf{n}_1 \times \mathbf{n}_2||}. \tag{2.13}$$

To implement this, we need to distinguish two waves: shock and shear waves. One possibility is to look at the dot-product of $\hat{\Delta}\vec{q}$ and $\nabla\rho/||\nabla\rho||$: shock if close to 1 and shear if close to 0. Then,

$$\mathbf{n}_1 = \begin{cases} \hat{\Delta}\vec{q}, & \text{if } ||\Delta\vec{q}|| > \epsilon, \\ \mathbf{n}_\perp, & \text{otherwise,} \end{cases}, \tag{2.14}$$

$$\mathbf{n}_2 = \begin{cases} \mathbf{n}, & \text{if } ||\nabla\vec{q}|| < \epsilon, \\ \dfrac{(\mathbf{n}_1 \times \mathbf{n}) \times \mathbf{n}_1}{||(\mathbf{n}_1 \times \mathbf{n}) \times \mathbf{n}_1||}, & \text{else if } ||\nabla\rho|| < \epsilon \\ \dfrac{\vec{q}_L - (\vec{q}_L \cdot \hat{\Delta}\vec{q})\hat{\Delta}\vec{q}}{||\vec{q}_L - (\vec{q}_L \cdot \hat{\Delta}\vec{q})\hat{\Delta}\vec{q}||}, & \text{else if } \hat{\Delta}\vec{q} \cdot \nabla\rho/||\nabla\rho|| \geq 0.5, \\ \dfrac{(\mathbf{n}_1 \times \nabla\rho) \times \mathbf{n}_1}{||(\mathbf{n}_1 \times \nabla\rho) \times \mathbf{n}_1||}, & \text{else if } \hat{\Delta}\vec{q} \cdot \nabla\rho/||\nabla\rho|| < 0.5, \end{cases}, \tag{2.15}$$

$$\mathbf{n}_3 = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{||\mathbf{n}_1 \times \mathbf{n}_2||}. \tag{2.16}$$

To save the computational cost, we may skip all computations associated with zero coefficients.

References

[1] H. Nishikawa and K. Kitamura. Very simple, carbuncle-free, boundary-layer resolving, rotated-hybrid Riemann solvers. *J. Comput. Phys.*, 227:2560–2581, 2007.