Integration Formulas on Triangle

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1 Notations

Consider a triangle, T, having vertices 1, 2, and 3 ordered counterclockwise. The vertex coordinates are denoted by (x_1, y_1), (x_2, y_2), and (x_3, y_3). The centroid coordinates are denoted by $(x_c, y_c) = ((x_1 + x_2 + x_3)/3, (y_1 + y_2 + y_3)/3)$. The difference operator, $\Delta()_i$, is defined by

$$\Delta x_1 = x_2 - x_3, \quad \Delta x_2 = x_3 - x_1, \quad \Delta x_3 = x_1 - x_2, \tag{1}$$

and similarly for y. The centroid difference operator, $\Delta()_{ic}$, is defined by

$$\Delta x_{ic} = x_i - x_c, \tag{2}$$

for $i = 1, 2, 3$, and similarly for y. The triangle area is denoted by S_T.

2 Integration Formulas I

We consider integrations in the following form:

$$I(m,n) = \frac{1}{2S_T}\int_T (x - x_c)^m\,(y - y_c)^n\,dx\,dy \tag{3}$$

where m and n are integers. It can be evaluated as follows:

$$I(1,0) = I(0,1) = 0 \tag{4}$$

$$I(2,0) = \frac{1}{36}\left(\Delta x_3^2 + \Delta x_2\Delta x_3 + \Delta x_2^2\right) = \frac{1}{72}\sum_{i=1}^{3}\Delta x_i^2 = \frac{1}{24}\sum_{i=1}^{3}\Delta x_{ic}^2 \tag{5}$$

$$I(3,0) = \frac{1}{20}\Delta x_{1c}\Delta x_{2c}\Delta x_{3c} = \frac{1}{20}\prod_{i=1}^{3}\Delta x_{ic} \tag{6}$$

$$I(4,0) = \frac{1}{270}\{36I(2,0)\}^2 = \frac{1}{120}\left\{\sum_{i=1}^{3}\Delta x_{ic}^2\right\}^2 \tag{7}$$

$$I(5,0) = \frac{1}{63}\Delta x_{1c}\Delta x_{2c}\Delta x_{3c}\{36I(2,0)\} = \frac{1}{42}\prod_{i=1}^{3}\Delta x_{ic}\sum_{i=1}^{3}\Delta x_{ic}^2 \tag{8}$$

similarly for $I(0,m)$, $m = 2, 3, 4, 5$.

$$I(1,1) = \frac{1}{72}\left\{(2\Delta y_3 + \Delta y_2)\,\Delta x_3 + (2\Delta y_2 + \Delta y_3)\,\Delta x_2\right\} = \frac{1}{72}\sum_{i=1}^{3}\Delta x_i\Delta y_i = \frac{1}{24}\sum_{i=1}^{3}\Delta x_{ic}\Delta y_{ic} \tag{9}$$

$$I(2,1) = \frac{1}{540}\left\{-(2\Delta y_3 + \Delta y_2)\,\Delta x_3^2 + 2\,(\Delta y_2 + \Delta y_3)\,\Delta x_2\Delta x_3 + (2\Delta y_2 + \Delta y_3)\,\Delta x_2^2\right\}$$

$$= -\frac{1}{180}\sum_{i=1}^{3}\Delta x_i \Delta x_{ic}\Delta y_i = \frac{1}{60}\sum_{i=1}^{3}\Delta x_{ic}^2 \Delta y_{ic} \quad (10)$$

$$\begin{aligned}
I(3,1) &= \frac{1}{540}\{36I(2,0)\}\{(\Delta y_2 + 2\Delta y_3)\,\Delta x_3 + (\Delta y_3 + 2\Delta y_2)\,\Delta x_2\} \\
&= \frac{1}{540}\left\{(\Delta y_2 + 2\Delta y_3)\,\Delta x_3^3 + 3\,(\Delta y_2 + \Delta y_3)\,\Delta x_2 \Delta x_3^2 + 3\,(\Delta y_2 + \Delta y_3)\,\Delta x_2^2 \Delta x_3 + (\Delta y_3 + 2\Delta y_2)\,\Delta x_2^3\right\} \\
&= \frac{1}{5} I(2,0)\sum_{i=1}^{3}\Delta x_{ic}\Delta y_{ic} = \frac{1}{120}\sum_{i=1}^{3}\Delta x_{ic}^2 \sum_{i=1}^{3}\Delta x_{ic}\Delta y_{ic} \quad (11)
\end{aligned}$$

$$\begin{aligned}
I(4,1) &= \frac{1}{8505}\{5\,(2\Delta y_2 + \Delta y_3)\,\Delta x_2^4 + 4\,(5\Delta y_2 + \Delta y_3)\,\Delta x_2^3 \Delta x_3 + 6\,(\Delta y_2 - \Delta y_3)\,\Delta x_2^2 \Delta x_3^2 \\
&\quad -4\,(5\Delta y_3 + \Delta y_2)\,\Delta x_2 \Delta x_3^3 - 5\,(2\Delta y_3 + \Delta y_2)\,\Delta x_3^4\} \\
&= \frac{1}{2100}\left\{2\prod_{i=1}^{3}\Delta x_{ic}\sum_{i=1}^{3}\Delta x_{ic}\Delta y_{ic} + \sum_{i=1}^{3}\Delta x_{ic}^2 \sum_{i=1}^{3}\Delta x_{ic}^2 \Delta y_{ic}\right\} \quad (12)
\end{aligned}$$

(13)

$$\begin{aligned}
I(2,2) &= \frac{1}{540}\left\{\left(\Delta x_3^2 + (\Delta x_2 + \Delta x_3)^2\right)\Delta y_3^2 + 2\,(\Delta x_2 + \Delta x_3)^2\,\Delta y_2 \Delta y_3 + \left(\Delta x_2^2 + (\Delta x_2 + \Delta x_3)^2\right)\Delta y_2^2\right\} \\
&= \frac{1}{540}\sum_{i=1}^{3}\Delta x_i^2 \Delta y_i^2 = \frac{1}{180}\sum_{i=1}^{3}\Delta x_{ic}^2 \Delta y_{ic}^2 \quad (14)
\end{aligned}$$

$$\begin{aligned}
I(3,2) &= \frac{4}{540\cdot 63}\{\left(10\Delta x_2^2 + 10\Delta x_2 \Delta x_3 + \Delta x_3^2\right)\Delta y_2^3 + \left(15\Delta x_2^2 - 3\Delta x_3^2 + 6\Delta x_3 \Delta x_2\right)\Delta y_3 \Delta y_2^2 \\
&\quad - \left(15\Delta x_3^2 - 3\Delta x_2^2 + 6\Delta x_3 \Delta x_2\right)\Delta y_3^2 \Delta y_2 - \left(10\Delta x_3^2 + 10\Delta x_2 \Delta x_3 + \Delta x_2^2\right)\Delta y_3^3\} \\
&= \frac{1}{4200}\left\{\prod_{i=1}^{3}\Delta x_{ic}\sum_{i=1}^{3}\Delta y_{ic}^2 + \sum_{i=1}^{3}\Delta x_{ic}^2 \sum_{i=1}^{3}\Delta x_{ic}\Delta y_{ic}^2 + 2\sum_{i=1}^{3}\Delta x_{ic}^2 \Delta y_{ic}\sum_{i=1}^{3}\Delta x_{ic}\Delta y_{ic}\right\} \quad (15)
\end{aligned}$$

3 Integration Formulas II

Next consider

$$I_2(n) = \frac{1}{2S_T}\int_T \{a\,(x - x_c) + b\,(y - y_c)\}^n \, dx\, dy \quad (16)$$

It can be evaluated as follows:

$$\begin{aligned}
I_2(0) &= \frac{1}{2} \quad (17) \\
I_2(1) &= 0 \quad (18) \\
I_2(2) &= a^2 I(2,0) + 2abI(1,1) + b^2 I(0,2) = \frac{1}{24}\sum_{i=1}^{3}\left\{a^2 \Delta x_{ic}^2 + 2ab\Delta x_{ic}\Delta y_{ic} + b^2 \Delta y_{ic}^2\right\} \\
&= \frac{1}{24}\sum_{i=1}^{3}(a\Delta x_{ic} + b\Delta y_{ic})^2 = \frac{1}{72}\sum_{i=1}^{3}(a\Delta x_i + b\Delta y_i)^2 \\
&= \frac{1}{36}\{b^2\left(\Delta y_3^2 + \Delta y_2 \Delta y_3 + \Delta y_2^2\right) + ab\,(\Delta x_2 \Delta y_3 + \Delta x_3 \Delta y_2 + 2\Delta x_2 \Delta y_2 + 2\Delta x_3 \Delta y_3) \\
&+ a^2\left(\Delta x_3^2 + \Delta x_2 \Delta x_3 + \Delta x_2^2\right)\} \quad (19) \\
I_2(3) &= a^3 I(3,0) + 3a^2 bI(2,1) + 3ab^2 I(1,2) + b^3 I(0,3) \\
&= -\frac{1}{15*36}\{b\,(\Delta y_3 + 2\Delta y_2) + a\,(\Delta x_3 + 2\Delta x_2)\}\{b\,(\Delta y_3 - \Delta y_2) + a\,(\Delta x_3 - \Delta x_2)\}
\end{aligned}$$

$$
\begin{aligned}
& \{b(\Delta y_2 + 2\Delta y_3) + a(\Delta x_2 + 2\Delta x_3)\} \\
&= \frac{1}{20}(a\Delta x_{3c} + b\Delta y_{3c})(a\Delta x_{1c} + b\Delta y_{1c})(a\Delta x_{2c} + b\Delta y_{2c}) \\
&= \frac{1}{20}\prod_{i=1}^{3}(a\Delta x_{ic} + b\Delta y_{ic}) \\
&= \frac{1}{20}\left(a^3\prod_{i=1}^{3}\Delta x_{ic} + a^2 b\sum_{i=1}^{3}\Delta x_{ic}^2\Delta y_{ic} + ab^2\sum_{i=1}^{3}\Delta x_{ic}\Delta y_{ic}^2 + b^3\prod_{i=1}^{3}\Delta y_{ic}\right) \qquad (20)
\end{aligned}
$$

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\begin{aligned}
I_2(4) &= \frac{2}{15*36}\{36I_2(2)\}^2 = \frac{24}{5}\{I_2(2)\}^2 = \frac{1}{120}\left\{\sum_{i=1}^{3}(a\Delta x_{ic} + b\Delta y_{ic})^2\right\}^2 \\
&= a^4 I(4,0) + 4a^3 b I(3,1) + 6a^2 b^2 I(2,2) + 4ab^3 I(1,3) + b^4 I(0,4) \qquad (21)
\end{aligned}
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\begin{aligned}
I_2(5) &= \frac{60}{189*36}\{36I_2(3)\}\{36I_2(2)\} = -\frac{4}{189*36}\{-540I_2(3)\}\{36I_2(2)\} \\
&= \frac{2160}{189}\{I_2(3)\}\{I_2(2)\} \\
&= \frac{80}{7}\frac{1}{20}\prod_{i=1}^{3}(a\Delta x_{ic} + b\Delta y_{ic})\frac{1}{24}\sum_{i=1}^{3}(a\Delta x_{ic} + b\Delta y_{ic})^2 \\
&= \frac{1}{42}\prod_{i=1}^{3}(a\Delta x_{ic} + b\Delta y_{ic})\sum_{i=1}^{3}(a\Delta x_{ic} + b\Delta y_{ic})^2 \\
&= a^5 I(5,0) + 5a^4 b I(4,1) + 10a^3 b^2 I(3,2) + 10a^2 b^3 I(2,3) + 5ab^4 I(1,4) + b^5 I(0,5) \qquad (22)
\end{aligned}
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(23)