A Difficulty of Computing a Potential Vortex by Residual Minimization

Hiroaki Nishikawa

March, 2000

1 Triangular Grids

Consider numerically solving Cauchy-Riemann equations for velocity components, u and v in a domain bounded by two co-centric circles, by the least-square residual minimization scheme on *triangular grids*. As boundary conditions, we give a constant flow speed q_∞ at the outer boundary and impose $u_n = 0$ at the inner boundary. Under these conditions, the correct solution is a potential vortex.

$$q = \frac{k}{r} \tag{1}$$

where $q^2 = u^2 + v^2$, $r^2 = x^2 + y^2$, and k is a constant related to the circulation. However, numerical experiments show that the least-squares solution is extremely poor, and that the method is not even first-order accurate for this particular problem. It was observed that the method tried to compute something different from the true solution. It is a solid body rotaion.

$$q = c\,r \tag{2}$$

where c is a constant which is actually equal to a half of the vorticity. This solution does not satisfy one of the Cauchy-Riemann equations, i.e. irrotationality condition, but does satisfy Laplace equations for u and v. Because the least-squares method is equivalent to Galerkin finite element method for the associated Laplace equations, it seems that the method with the above boundary conditions do not know which solution to compute, and therefore trying to compute a smooth combination of the two. Yet another interesting observation is that u and v in the wrong solution are linear in x and y which are represented exactly by piecewise linear elements such as triangles whereas those in the correct solution are not.

To investigate this problem in more detail, let us examine the truncation error. The residuals that we minimize are

$$\Delta_T = \frac{1}{2S_T}\sum_{i\in j_T}(u_i\Delta y_i - v_i\Delta x_i) \tag{3}$$

$$\Omega_T = \frac{1}{2S_T}\sum_{i\in j_T}(u_i\Delta x_i + v_i\Delta y_i) \tag{4}$$

for an arbitrary triangle T. For simplicity, we consider a grid with uniform $\Delta\theta$ a part of which is shown in Figure 1. We first consider a triangle of type 1. Substituting the correct solution (1) into the residuals, we obtain

$$\Delta_1 = -\frac{k(2r+\Delta r)}{r(r+\Delta r)^2}\tan\left(\frac{\Delta\theta}{2}\right) \tag{5}$$

$$\Omega_1 = -\frac{k\Delta r}{r(r+\Delta r)^2}. \tag{6}$$

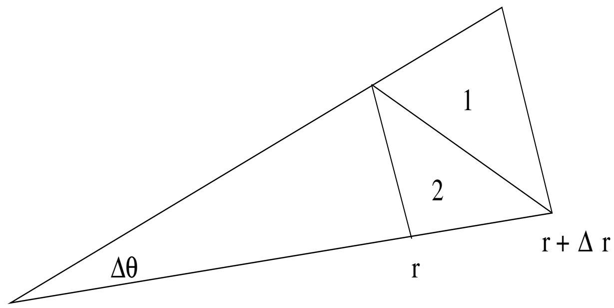

Figure 1: A pair of triangular cells in a polar grid.

Hence the exact solution does not satisfy the discrete equations exactly, and therefore these represent the truncation errors, i.e. the minimum values that can be attained. On the other hand, substitution of (2) yields

$$\Delta_1 = 0 \tag{7}$$

$$\Omega_1 = 2\,c, \tag{8}$$

i.e. it satisfies the discrete relations exactly as expected. We now realize that the minimization scheme would prefer the wrong solution at least in terms of Δ_T because the residual can be smaller with wrong solution. But, as will be shown later, our numerical experiments show that the problem comes from Ω_T rather than Δ_T, by the fact that the global condition of constant circulation is violated more seriously than the global continuity condition. We thus focus our attention to the vorticity residuals. Since it is not clear for which solution it is smaller, let us consider the difference of the vorticity residuals, (6) and (8),

$$2|c| - \frac{|k|\Delta r}{r(r+\Delta r)^2} \tag{9}$$

where we have put the absolute sign so that both quantities are positive, and also that the analysis will be valid for the other triangle. Keep in mind that if this is positive, we expect that the least-squares method finds the correct solution, but if negative, the wrong solution may be computed instead because of its smaller residual value. We first eliminate c using the fact that the two solutions must match at the outer boundary, say at $r = R$,

$$\frac{k}{R} = cR, \tag{10}$$

by which (9) simplifies to

$$|k|\left(\frac{2r(r+\Delta r)^2 - R^2\Delta r}{rR^2(r+\Delta r)^2}\right). \tag{11}$$

Introducing the nondimensional variables, $\tilde{r} = r/R$ and $\Delta\tilde{r} = \Delta r/R$, we have

$$\frac{|k|}{R^2}\left(\frac{2\tilde{r}(\tilde{r}+\Delta\tilde{r})^2 - \Delta\tilde{r}}{\tilde{r}(\tilde{r}+\Delta\tilde{r})^2}\right). \tag{12}$$

We are then interested in the sign of the numerator. To investigate this, we define

$$f = 2\tilde{r}(\tilde{r}+\Delta\tilde{r})^2 - \Delta\tilde{r}, \tag{13}$$

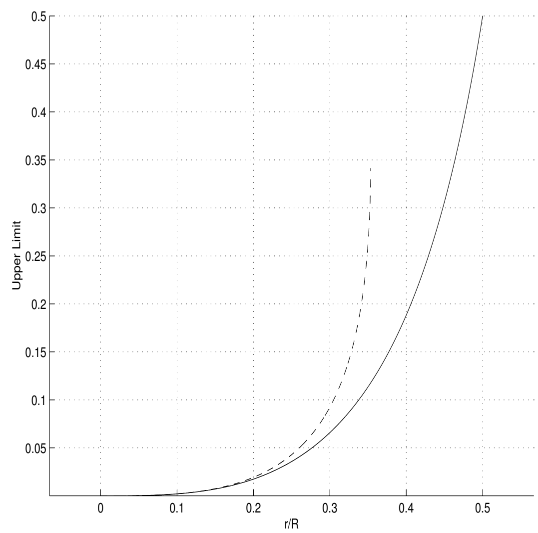

Figure 2: The upper limit of $\Delta\tilde{r}$ vs $\tilde{r}$ in the first inequality in (17).

Considering this equation as quadratic in $\Delta\tilde{r}$, we find the determinant

$$D = 1 - 8\tilde{r}^2. \tag{14}$$

If this is positive, i.e. if

$$\tilde{r} \leq \frac{\sqrt{2}}{4} \approx 0.35, \tag{15}$$

$f(\Delta\tilde{r}, \tilde{r})$ is negative for some $\Delta\tilde{r}$. Hence, there is a possibility that the residual can be smaller for the wrong solution near the inner boundary. But the question remains as to for what value of $\Delta\tilde{r}$ this happens since it might happen where the values of $\Delta\tilde{r}$ are totally impractical. Directly solving $f = 0$ for $\Delta\tilde{r}$, we obtain

$$\Delta\tilde{r} = \frac{(1-4\tilde{r}^2) \pm \sqrt{1-8\tilde{r}^2}}{4\tilde{r}} \tag{16}$$

For $1 - 8\tilde{r}^2 > 0$, it can be easily shown that the solutions are both positive. Therefore, for the residual to be smaller with the true solution, we must have

$$\Delta\tilde{r} < \frac{(1-4\tilde{r}^2) - \sqrt{1-8\tilde{r}^2}}{4\tilde{r}} \quad \text{or} \quad \frac{(1-4\tilde{r}^2) + \sqrt{1-8\tilde{r}^2}}{4\tilde{r}} < \Delta\tilde{r}. \tag{17}$$

The lower limit on the right increases as $\tilde{r}$ decreases with the minimum approximately 0.35, which means that we need to use larger values of $\Delta\tilde{r}$ as we approach the inner boundary. For example, $\Delta\theta$ must be greater than 4.8 at $\tilde{r} = 0.1$. This is meaningless. So, we focus on the first inequality from which we see that the upper limit decreases as $\tilde{r}$ decreases with the maximum approximately 0.35 (See Figure 2). From the figure we see that the grid must get finer and finer as we approach the inner boundary which is typically located at $\tilde{r} = 0.05$, and that the cell size must be below the curve which requires very fine grid in the vicinity of the inner boundary. For example, the upper limit is 2.5E-04 at $\tilde{r} = 0.05$ and $2.1E-03$ at $\tilde{r} = 0.1$. By an actual computation, we have found that there must be approximately 100 elements in the radial direction between $\tilde{r} = 0.05$ and $\tilde{r} = 0.35$ in order to meet the condition (17).

We now consider a triangle of type 2. In this case, we have for the correct solution,

$$\Delta_2 = -\frac{k(2r+\Delta r)}{r^2(r+\Delta r)} \tan\left(\frac{\Delta\theta}{2}\right) \tag{18}$$

$$\Omega_2 = \frac{k\Delta r}{r^2(r+\Delta r)}. \tag{19}$$

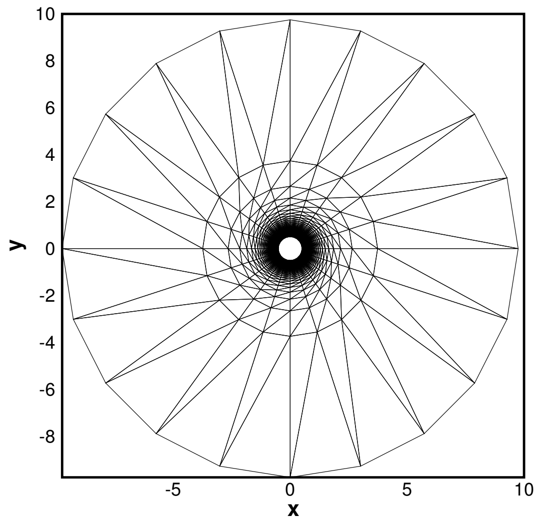

Figure 3: A grid with the spacing two times the upper limit. 50x20 grid with 2000 triangles and 1020 nodes.

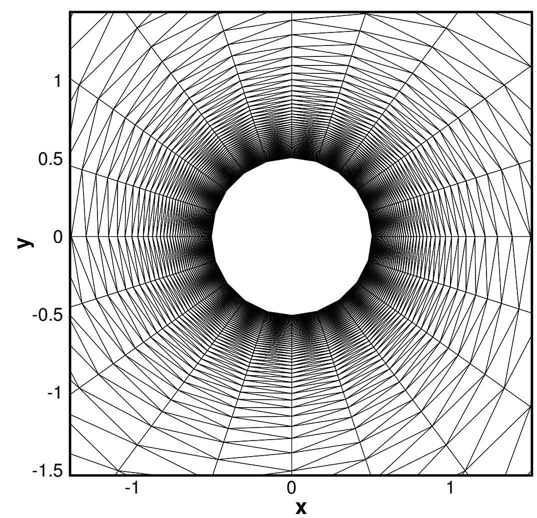

Figure 4: A close view of the grid in Figure 1. The minimum spacing in the radial direction is 5.0E-03.

We again consider the vorticity residual only. Similarly to the previous analysis, the difference between this truncation error and the wrong vorticity can be written as

$$\frac{|k|}{R^2}\left(\frac{2\tilde{r}^2(\tilde{r}+\Delta\tilde{r})-\Delta\tilde{r}}{\tilde{r}(\tilde{r}+\Delta\tilde{r})^2}\right). \tag{20}$$

We then define

$$f = 2\tilde{r}^2(\tilde{r}+\Delta\tilde{r})-\Delta\tilde{r} = \Delta\tilde{r}(2\tilde{r}^2-1)+2\tilde{r}^3. \tag{21}$$

It is easy to show that f is positive if

$$\Delta\tilde{r} < \frac{2\tilde{r}^3}{1-2\tilde{r}^2} \tag{22}$$

for $\tilde{r} < \frac{1}{\sqrt{2}} \approx 0.7$. This upper limit is shown as a sold curve in Figure 2. Although this curve is located below the other one, the difference is not so large. In fact, for small $\tilde{r}$, we have

$$\frac{(1-4\tilde{r}^2)-\sqrt{1-8\tilde{r}^2}}{4\tilde{r}} \approx 2\tilde{r}^3+8\tilde{r}^5 \tag{23}$$

$$\frac{2\tilde{r}^3}{1-2\tilde{r}^2} \approx 2\tilde{r}^3+4\tilde{r}^5. \tag{24}$$

A computation with a grid that satisfies the condition has verified our analysis, producing no appearance of the wrong solution and giving a very accurate solution that preserves the circulation perfectly. The grid was a very fine grid generated by advancing outward from the inner boundar at $r = 0.5$ up to the outer boundary at $r \approx 10.0$ using the formula for the upper limit for type 1 triangles (but, beyond $\tilde{r} \approx 0.35$, the formula gives imaginary numbers, so 1.7 times the last value of $\Delta\tilde{r}$ was used thereafter) with $\Delta\theta = \frac{2\pi}{20}$, resulting 3920 triangles and 1980 nodes (98x20 grid). Note that this grid has spacings exceeding the values given by the solid curve for some $\tilde{r}$. This implies that we do not have to satisfy the condition so strictly.

Another computation was made on a grid generated similarly, but by using twice the maximum $\Delta\tilde{r}$, which is still a very fine grid as can be seen in Figure 3 and 4. It was found that the method significantly underestimated the circulation along the inner boundary, 1.42 against the exact value 3.09, implying that a significant lift reduction would be expected for general lifting flow problems which necessarily contain a point vortex solution.

Also we performed a computation on a coarse grid with the spacing 10 times the upper limit which has 480 triangles and 260 nodes which is perhaps of a more practical size (12x20 grid). In this case, the computed circulation at inner boundary was found to be 0.206 which is extremely inaccurate.

It has been also found that minimizing vorticity residual only at the inner boundary improves the solution. At least, for nonlifting cases (zero circulation), the procedure is very effective, but not very much for lifting cases. When applied to the coarse grid case, the method produced the circulation 0.511 which is again extremely inaccurate.

Yet another grid was tested which was generated from the coarse grid by increasing the elements in the circumferencial direction to 160 in order to investigate the effect of the divergence residual. We obtained the circulation 0.516 against the exact value 3.14 which is better than the previous ones but not enough at all. This verifies that the problem is originated from the vorticity residual.

2 Quadrilateral Grids

For quadrilateral grids, the least-squares method is very accurate. This is because the correct solution satisfies the discrete equations exactly. The residuals are in fact equal to the area-weighted sum of the residuals of the two triangles forming a quadrilateral. It can be easily found that for the quadrilateral Q formed by the pair of triangles in Figure 1,

$$\Delta_Q = 0 \quad (25)$$
$$\Omega_Q = 0 \quad (26)$$

for the correct solution, and

$$\Delta_Q = 0 \quad (27)$$
$$\Omega_Q = 2c \quad (28)$$

for the wrong solution. Therefore, the method finds the correct solution which gives smaller residuals.

3 Concluding Remarks

It has been shown that the residual minimization scheme requires a higly stretched grid spacing in the radial direction near the inner boundary up to $r/R \approx 0.35$ in order to compute flows with nonzero circulation. An analytical formula for the maximum spacing in that region has been derived. Numerical experiments have shown that violating the rule can yield significant amount of circulation reduction along the inner body, and also that the problem is mainly related to the vorticity residual.

The grid that meets the condition was found to be too fine to be used in practice, which looks like the one for viscous flow simulations. Some way to circumvent this difficulty is sought. Considering the fact that the method is very accurate if the exact solution is specified also along the inner boundary, we may hope that the problem could be solved by devising a new boundary procedure. It seems however still difficult.