6.5 Exact Solutions for Linear Systems of Conservation Laws

Consider the linear hyperbolic system of one-dimensional conservation laws of the form,

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = 0. \tag{6.5.1}$$

where $\mathbf{U} = \mathbf{U}(x,t)$ and $\mathbf{A}$ is a constant coefficient matrix having real eigenvalues and linearly independent eigenvectors. If we are given an initial solution as

$$\mathbf{U}(x,0) = \mathbf{U}_0(x), \tag{6.5.2}$$

then, we can find the exact solution as follows. First, we multiply (6.5.1) by the left-eigenvector matrix $\mathbf{L}$ (of $\mathbf{A}$) from the left,

$$\mathbf{L}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{LARL}\frac{\partial \mathbf{U}}{\partial x} \quad = \quad 0, \tag{6.5.3}$$

where $\mathbf{R}$ is the right-eigenvector matrix (the inverse of $\mathbf{L}$). Because the matrix $\mathbf{L}$ is constant, we can write

$$\frac{\partial (\mathbf{LU})}{\partial t} + \mathbf{LAR}\frac{\partial (\mathbf{LU})}{\partial x} \quad = \quad 0. \tag{6.5.4}$$

Defining the characteristic variables (or the Riemann invariants) by

$$\mathbf{W} = \mathbf{LU}, \tag{6.5.5}$$

and the diagonal matrix $\mathbf{\Lambda}$ by $\mathbf{\Lambda} = \mathbf{LAR}$, we arrive at the diagonalized form of the system (6.5.1).

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{W}}{\partial x} = 0. \tag{6.5.6}$$

We point out that each component is now a linear scalar advection equation

$$\frac{\partial w_k}{\partial t} + \lambda_k\frac{\partial w_k}{\partial x} = 0, \tag{6.5.7}$$

where w_k denotes the k-th component of $\mathbf{W}$ and λ_k is the k-th eigenvalue. Therefore, the exact solution of the k-th component is given simply by

$$w_k(x,t) = w_k(x - \lambda_k t, 0). \tag{6.5.8}$$

Then, by definition (6.5.5), we obtain

$$\mathbf{U}(x,t) = \mathbf{RW}(x,t) = \sum_k w_k(x,t)\mathbf{r}_k = \sum_k w_k(x-\lambda_k t, 0)\mathbf{r}_k, \tag{6.5.9}$$

or equivalently, (by (6.5.5) again),

$$\mathbf{U}(x,t) = \sum_k (\ell_k \mathbf{U}_0(x - \lambda_k t))\mathbf{r}_k, \tag{6.5.10}$$

where ℓ_k is the k-th row of $\mathbf{L}$ (the k-th left-eigenvector) and $\mathbf{r}_k$ is the k-th column of $\mathbf{R}$ (the k-th right-eigenvector). This is the exact solution for the initial solution (6.5.2).

Now we consider a simple wave. A simple wave is defined as a solution whose variation is confined in a one-dimensional subspace spanned by a single eigenvector. For example, if the initial solution is a j-th simple wave, we project it onto the j-th eigenspace to write

$$\mathbf{U}_0(x) = \sum_k (\ell_k \mathbf{U}_0(x))\mathbf{r}_k \tag{6.5.11}$$

$$= (\ell_j \mathbf{U}_0(x))\mathbf{r}_j + \sum_{k:k\neq j} c_k \mathbf{r}_k, \tag{6.5.12}$$

where c_k are constant. Then, the exact solution (6.5.10) can be written as

$$\mathbf{U}(x,t) = \sum_k (\ell_k \mathbf{U}_0(x-\lambda_j t))\mathbf{r}_k \tag{6.5.13}$$

$$= (\ell_j \mathbf{U}_0(x-\lambda_j t))\mathbf{r}_j + \sum_{k:k\neq j} c_k \mathbf{r}_k \tag{6.5.14}$$

$$= \mathbf{U}_0(x-\lambda_j t). \tag{6.5.15}$$

Therefore, the initial solution is preserved perfectly for a simple wave. This is natural because only one advection speed is relevant in a simple wave. Finally, we give an example: set the following initial solution,

$$\mathbf{U}_0(x) = \overline{\mathbf{U}} + \sigma \sin(x)\, \mathbf{r}_j, \tag{6.5.16}$$

where $\overline{\mathbf{U}}$ is a constant state and σ is an amplitude of the sine wave, then the exact solution is given by

$$\mathbf{U}(x,t) = \mathbf{U}_0(x-\lambda_j t) = \overline{\mathbf{U}} + \sigma \sin(x-\lambda_j t)\, \mathbf{r}_j. \tag{6.5.17}$$

It is indeed simple and also general enough to be applicable to arbitrary linear hyperbolic systems of the form (6.5.1). Particularly, I like the fact that the result (and all the above discussion) is valid even for steady systems, i.e., (6.5.1) with (x,t) replaced by (x,y) or (y,x). This is very nice.

6.6 Simple Wave Solutions for Nonlinear Systems of Conservation Laws

Consider the nonlinear hyperbolic system of one-dimensional conservation laws of the form,

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0. \tag{6.6.1}$$

or

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = 0. \tag{6.6.2}$$

where $\mathbf{U} = \mathbf{U}(x,t)$ and $\mathbf{A}(\mathbf{U})$ is a coefficient matrix which is diagonalizable but no longer constant. Unlike the linear case, for a given initial solution such as

$$\mathbf{U}(x,0) = \mathbf{U}_0(x), \tag{6.6.3}$$

we cannot find the exact solution generally in a closed form. To see this, as in the linear case, we multiply (6.6.2) by the left-eigenvector matrix $\mathbf{L}$ from the left,

$$\mathbf{L}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{LARL}\frac{\partial \mathbf{U}}{\partial x} = 0. \tag{6.6.4}$$

Here is a difficulty. We can no longer define the characteristic variables as in (6.5.5) because $\mathbf{L}$ is not constant any more. Instead, I define the characteristic variables by

$$\partial \mathbf{W} = \mathbf{L}\partial \mathbf{U}, \tag{6.6.5}$$

and arrive at the diagonalized form of the system.

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{W}}{\partial x} = 0. \tag{6.6.6}$$

Each component is now a nonlinear scalar advection equation with the exact solution,

$$w_k(x,t) = w_k(x - \lambda_k t, 0), \tag{6.6.7}$$

which is valid up to the time when a shock forms. Note that this solution is given implicitly in general because λ_k depends on the solution. Then, we use (6.6.5) to get the solution in the conservative variables, (6.6.5)

$$\partial \mathbf{U}(x,t) = \mathbf{R}\partial \mathbf{W}(x,t) = \sum_k \partial w_k(x,t)\mathbf{r}_k = \sum_k \partial w_k(x - \lambda_k t, 0)\mathbf{r}_k, \tag{6.6.8}$$

and so

$$\partial \mathbf{U}(x,t) = \sum_k (\ell_k \partial \mathbf{U}_0(x - \lambda_k t))\mathbf{r}_k. \tag{6.6.9}$$

I like it, but it is too bad that we cannot get $\mathbf{U}(x,t)$ unless this is integrable. We cannot project even the initial solution onto the space of an eigenvector exactly:

$$\mathbf{U}_0(x) \quad \neq \quad \sum_k (\ell_k \mathbf{U}_0(x))\mathbf{r}_k. \tag{6.6.10}$$

Then, we consider a simple wave solution because it gives us a hope to get exact solutions. Let $h(x,t)$ be any smooth function that satisfies the j-th component of (6.6.6),

$$\frac{\partial h}{\partial t} + \lambda_j(\mathbf{U})\,\frac{\partial h}{\partial x} \quad = \quad 0, \tag{6.6.11}$$

so that $h(x,t)$ is the j-th characteristic variable $w_j(x,t) = h(x - \lambda_j(\mathbf{U})\,t) = h(\xi)$, and assume that other characteristic variables are constant. Then, we have from (6.6.5)

$$\partial \mathbf{U}(x,t) = \mathbf{r}_j(\mathbf{U})\,\partial h(\xi), \tag{6.6.12}$$

or because $\mathbf{U}(x,t) = \mathbf{U}(\xi)$, we may write

$$\frac{d\mathbf{U}(\xi)}{d\xi} = \mathbf{r}_j(\mathbf{U})\,\frac{dh(\xi)}{d\xi}. \tag{6.6.13}$$

If this is integrable, we can find an exact solution $\mathbf{U}$ for the j-th simple wave. This is possible in some cases. Note that the variable $\mathbf{U}$ can be taken to be any variable (e.g., conservative or primitive) as long as the eigenvector $\mathbf{r}_j$ is defined consistently. Some examples are given in the next section.

6.7 Some Exact Simple Wave Solutions

6.7.1 Case 1: $\mathbf{r}_j$ = constant

I like eigenvectors that are independent of the solution because the ordinary differential equation (6.6.13) is then trivial. We readily obtain a simple wave solution as

$$\mathbf{U}(\xi) = h(\xi)\,\mathbf{r}_j + \text{constant}. \tag{6.7.1}$$

The entropy wave in the Euler equations is a good example: the eigenvector, based on the primitive variables $\mathbf{V} = [\rho, u, p]^t$, is given by

$$\mathbf{r}_j = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{6.7.2}$$

with eigenvalue $\lambda_j = u$. So, it follows immediately from (6.7.1) that

$$\rho \quad = \quad h(x - u\,t) + \text{constant}, \tag{6.7.3}$$

$$u \quad = \quad \text{constant}, \tag{6.7.4}$$

$$p \quad = \quad \text{constant}. \tag{6.7.5}$$

This shows that the wave does not deform because the speed u is constant (the entropy wave is linearly degenerate). Any function (even a discontinuous one) can be chosen for $h(x)$. It will be simply convected at the velocity u.

6.7.2 Case 2: $\mathbf{r}_j \propto \mathbf{U}$

Consider the case where the eigenvector is proportional to the solution vector,

$$\mathbf{r}_j = \overline{\mathbf{M}}\,\mathbf{U}, \tag{6.7.6}$$

where $\overline{\mathbf{M}}$ is a constant matrix. In this case, we have from (6.6.13),

$$\frac{d\mathbf{U}(\xi)}{d\xi} = \overline{\mathbf{M}}\,\mathbf{U}\frac{dh(\xi)}{d\xi} = \overline{\mathbf{M}}\,\mathbf{U}h^{'}(\xi). \tag{6.7.7}$$

This is a linear system of ordinary differential equations and therefore can be solved analytically by a standard technique. This is nice. A good example is the Alfvén wave of the ideal magnetohydrodynamic equations. For the primitive variables $\mathbf{U} = [\rho, v_x, v_y, v_z, B_x, B_y, B_z, p]^t$, where (v_x, v_y, v_z) is the velocity vector and (B_x, B_y, B_z) is the magnetic field, with the coefficient matrix,

$$\mathbf{A} = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & \dfrac{B_y}{4\pi\rho} & \dfrac{B_z}{4\pi\rho} & \dfrac{1}{\rho} \\ 0 & 0 & v_x & 0 & -\dfrac{B_x}{4\pi\rho} & 0 & 0 \\ 0 & 0 & 0 & v_x & 0 & -\dfrac{B_x}{4\pi\rho} & 0 \\ 0 & B_y & -B_x & 0 & v_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & v_x & 0 \\ 0 & \rho a^2 & 0 & 0 & 0 & 0 & v_x \end{bmatrix}, \tag{6.7.8}$$

where a is the speed of sound ($a^2 = \gamma p/\rho$), the eigenvector for the Alfvén waves is given by

$$\mathbf{r}_j^{\pm} = \begin{bmatrix} 0 \\ 0 \\ \pm B_z \\ \mp B_y \\ -\sqrt{4\pi\rho}\,B_z \\ \sqrt{4\pi\rho}\,B_y \\ 0 \end{bmatrix}, \tag{6.7.9}$$

with the eigenvalues

$$\lambda_j^{\pm} = v_x \pm \frac{B_x}{\sqrt{4\pi\rho}}. \tag{6.7.10}$$

Note that zeroes in (6.7.9) correspond to the density, the velocity component v_x, and the pressure. This means that these variables are constant through the Alfvén waves (see Subsection 1.13.2). Therefore, the eigenvalues are constant; this wave is linearly degenerate. Now, because the density is constant, the eigenvector is linear in the primitive variable,

$$\mathbf{r}_j = \overline{\mathbf{M}}\,\mathbf{U}, \tag{6.7.11}$$

where $\mathbf{U}$ is taken to be the primitive variable and

$$\overline{\mathbf{M}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 & 0 & \mp 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{4\pi\rho} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{4\pi\rho} & 0 \end{bmatrix}. \tag{6.7.12}$$

Since only four variables, v_y, v_z, p, B_y, B_z, are relevant, we ignore other components in the following. Now, we derive an exact solution by solving (6.6.13):

$$\frac{d}{d\xi}\begin{bmatrix} v_y \\ v_z \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \pm 1 \\ 0 & 0 & \mp 1 & 0 \\ 0 & 0 & 0 & -\sqrt{4\pi\rho} \\ 0 & 0 & \sqrt{4\pi\rho} & 0 \end{bmatrix} \begin{bmatrix} v_y \\ v_z \\ B_y \\ B_z \end{bmatrix} h^{'}(\xi). \tag{6.7.13}$$

The standard diagonalization yields the decoupled system,

$$\frac{d(v_y \pm B_y/\sqrt{4\pi\rho})}{d\xi} = 0, \tag{6.7.14}$$

$$\frac{d(v_z \pm B_z/\sqrt{4\pi\rho})}{d\xi} = 0, \tag{6.7.15}$$

$$\frac{d(B_z + i\,B_y)}{d\xi} = -i\,a\,(B_z + i\,B_y)\,h^{'}(\xi), \tag{6.7.16}$$

$$\frac{d(B_z - i\,B_y)}{d\xi} = i\,a\,(B_z - i\,B_y)\,h^{'}(\xi), \tag{6.7.17}$$

whose general solution is

$$v_y(\xi) = \mp B_y(\xi)/\sqrt{4\pi\rho} + c_{vy}, \tag{6.7.18}$$

$$v_z(\xi) = \mp B_z(\xi)/\sqrt{4\pi\rho} + c_{vz}, \tag{6.7.19}$$

$$B_y(\xi) = K_R \cos(\sqrt{4\pi\rho}\,h(\xi)) + K_I \sin(\sqrt{4\pi\rho}\,h(\xi)), \tag{6.7.20}$$

$$B_z(\xi) = K_I \cos(\sqrt{4\pi\rho}\,h(\xi)) - K_R \sin(\sqrt{4\pi\rho}\,h(\xi)), \tag{6.7.21}$$

where c_{vy}, c_{vz}, K_R and K_I are arbitrary constants. This is the exact simple wave solution for the Alfvén wave in the most general form. Note that we can use only one solution, i.e., we can take only either positive or negative sign in the formula; otherwise it will not be a simple wave. Here is an example: take $h(\xi) = \xi/\sqrt{4\pi\rho}$ and $K_I = 0$, and obtain the simple wave associated with λ_j^+,

$$v_y(x - \lambda_j^+ t) = -K_R^{'} \cos(x - \lambda_j^+ t) + c_{vy}, \tag{6.7.22}$$

$$v_z(x - \lambda_j^+ t) = K_R^{'} \sin(x - \lambda_j^+ t) + c_{vz}, \tag{6.7.23}$$

$$B_y(x - \lambda_j^+ t) = \sqrt{4\pi\rho}\,K_R^{'} \cos(x - \lambda_j^+ t), \tag{6.7.24}$$

$$B_z(x - \lambda_j^+ t) = -\sqrt{4\pi\rho}\,K_R^{'} \sin(x - \lambda_j^+ t), \tag{6.7.25}$$

where c_{vy}, c_{vz}, and $K_R^{'}$ are arbitrary constants, and $K_R = \sqrt{4\pi\rho}\,K_R^{'}$. I like this simple Alfvén wave solution. It is linearly degenerate, and therefore the wave must be preserved at all times. This can be a good test case for code verification.

6.7.3 Case 3: Other cases

The acoustic wave in the Euler equations is a good example. In terms of the primitive variables, the acoustic eigenvectors are given by

$$\mathbf{r}_j^{\pm} = \begin{bmatrix} \pm\rho/a \\ 1 \\ \pm\rho a \end{bmatrix}, \tag{6.7.26}$$

where a is the speed of sound. The associated eigenvalues are

$$\lambda_j^{\pm} = u \pm a. \tag{6.7.27}$$

It must be kept in mind that there are two acoustic waves, $+$ and $-$, but we can take only either $+$ or $-$ for this solution to be a simple wave. Now, we insert these into (6.6.13) to get

$$\frac{d}{d\xi}\begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = \begin{bmatrix} \pm\rho/a \\ 1 \\ \pm\rho a \end{bmatrix} \frac{dh(\xi)}{d\xi}. \tag{6.7.28}$$

This is nonlinear, except for the velocity which can be integrated easily,

$$\int_{u_\infty}^{u} du = \int_{h_\infty}^{h} dh \tag{6.7.29}$$

to give

$$u(\xi) \quad = \quad h(\xi) + u_\infty, \tag{6.7.30}$$

where we have set $h_\infty = 0$ so that u_∞ is a mean flow value and any non-zero h can be thought of as a perturbation to the mean flow. Now, the density component can also be integrated if the flow is adiabatic, i.e. if

$$p/\rho^\gamma = p_\infty/\rho_\infty^\gamma. \tag{6.7.31}$$

We take advantage of this adiabatic relation to write the speed of sound in terms of the density as

$$\frac{a}{a_\infty} \quad = \quad \left(\frac{\rho}{\rho_\infty}\right)^{\frac{\gamma-1}{2}}, \tag{6.7.32}$$

which is substituted into the first component of (6.7.28) to yield

$$\frac{d\rho}{d\xi} \quad = \quad \pm\left(\frac{\rho}{\rho_\infty}\right)^{\frac{3-\gamma}{2}} \frac{\rho_\infty}{a_\infty}\frac{dh(\xi)}{d\xi}. \tag{6.7.33}$$

This can be integrated easily,

$$\int_1^{\rho/\rho_\infty} \left(\frac{\rho}{\rho_\infty}\right)^{\frac{\gamma-3}{2}} d\left(\frac{\rho}{\rho_\infty}\right) \quad = \quad \pm\frac{1}{a_\infty}\int_{h_\infty=0}^{h} dh, \tag{6.7.34}$$

resulting

$$\frac{\rho(\xi)}{\rho_\infty} \quad = \quad \left[1 \pm \frac{\gamma-1}{2}\frac{h(\xi)}{a_\infty}\right]^{\frac{2}{\gamma-1}}, \tag{6.7.35}$$

or by (6.7.30)

$$\frac{\rho(\xi)}{\rho_\infty} \quad = \quad \left[1 \pm \frac{\gamma-1}{2}\left(\frac{u(\xi)}{a_\infty} - M_\infty\right)\right]^{\frac{2}{\gamma-1}}, \tag{6.7.36}$$

where $M_\infty = u_\infty/a_\infty$. The pressure follows from the adiabatic relation (6.7.31),

$$\frac{p(\xi)}{p_\infty} \quad = \quad \left(\frac{\rho(\xi)}{\rho_\infty}\right)^\gamma = \left[1 \pm \frac{\gamma-1}{2}\left(\frac{u(\xi)}{a_\infty} - M_\infty\right)\right]^{\frac{2\gamma}{\gamma-1}}. \tag{6.7.37}$$

This completes the derivation of the exact simple wave solution for the acoustic wave. Now, we summarize the exact acoustic simple wave solution:

$$\frac{u(x-\lambda_j^\pm t)}{a_\infty} \quad = \quad M_\infty + \frac{h(x-\lambda_j^\pm t)}{a_\infty}, \tag{6.7.38}$$

$$\frac{\rho(x-\lambda_j^\pm t)}{\rho_\infty} \quad = \quad \left[1 \pm \frac{\gamma-1}{2}\left(\frac{u(x-\lambda_j^\pm t)}{a_\infty} - M_\infty\right)\right]^{\frac{2}{\gamma-1}}, \tag{6.7.39}$$

$$\frac{p(x-\lambda_j^\pm t)}{p_\infty} \quad = \quad \left[1 \pm \frac{\gamma-1}{2}\left(\frac{u(x-\lambda_j^\pm t)}{a_\infty} - M_\infty\right)\right]^{\frac{2\gamma}{\gamma-1}}, \tag{6.7.40}$$

where $\lambda_j^\pm = u \pm a$ and h is an arbitrary function which must be chosen to keep the density and the pressure positive. This is a very nice solution, but it is given implicitly, i.e., $\lambda_j^\pm$ depends on the solution itself, and also it is valid only when there are no shocks.

We point out here that the simple wave solution satisfies Burgers' equation as shown in Section 3.4.3. For example, the simple wave solution corresponding to $\lambda_j^+ = u + a$ satisfies (3.4.36) which is repeated here,

$$V_t + V\,V_x = 0, \tag{6.7.41}$$

where $V = u + a$. I like this because $V = u + a$ can be written in terms of u only, by using the fact that $u - \frac{2}{\gamma-1}a = u_\infty - \frac{2}{\gamma-1}a_\infty =$ constant:

$$V = \frac{\gamma+1}{2}u + \left(a_\infty - \frac{\gamma-1}{2}u_\infty\right), \tag{6.7.42}$$

so that, by solving this for u, a solution V of Burgers' equation (6.7.41) can be translated into the simple wave solution of the Euler equation by

$$\frac{u(\xi)}{a_\infty} = \frac{2}{(\gamma+1)}\frac{V(\xi)}{a_\infty} - \frac{2}{\gamma+1}\left(1 - \frac{\gamma-1}{2}M_\infty\right), \tag{6.7.43}$$

where $\xi = x - Vt$, and the density and the pressure follow from (6.7.39) and (6.7.40). Again, V must be chosen such that the density and the pressure stay positive. I like this solution because basically I can add any constant to u (and still V satisfies Burgers' equation), and so we may set up a solution as

$$\frac{u(\xi)}{a_\infty} = M_\infty + \frac{2}{(\gamma+1)}\frac{V(\xi)}{a_\infty}, \tag{6.7.44}$$

i.e., we use the solution V of Burgers' equation as a perturbation to the mean flow. Compare this with (6.7.38) and see how you can determine $h(\xi)$ by a solution $V(\xi)$ of Burgers' equation. See Subsection 7.11.1 for some examples.

The bottom-line is that the exact solution can be obtained if (6.6.5) is integrable so that the Riemann invariants can be defined explicitly. For example, the shallow-water equations and the isothermal Euler equations have this property. I like such equations.

6.8 Manufactured Solutions

Pick any function, substitute it into a governing equation you wish to solve, and define whatever left as a source term. This way, you can cook up an exact solution to any equations (by introducing a source term). It is so simple. I like it very much. Many people call such solutions manufactured solutions [83]. Here are some simple examples.

(a) 1D Linear Advection:

Substitute $u = \sin x$ into the advection equation (plus a possible source term f), $au_x = f$, to get

$$a\cos x = f, \tag{6.8.1}$$

which suggests to define $f = a\cos x$. Therefore, we can say that the advection equation,

$$au_x = f \quad \text{with} \quad f = a\cos x, \tag{6.8.2}$$

has the following exact solution,

$$u = \sin x. \tag{6.8.3}$$

(b) Poisson Equation:

Substitute $u = \sin x + \cos y$ into the Poisson equation, $u_{xx} + u_{yy} = f$, to get

$$-\sin x - \cos y = f, \tag{6.8.4}$$

which suggests to define $f = -\sin x - \cos y$. Therefore, the Poisson equation,

$$u_{xx} + u_{yy} = f \quad \text{with} \quad f = -\sin x - \cos y, \tag{6.8.5}$$

has the following exact solution,

$$u = \sin x + \cos y. \tag{6.8.6}$$

I like cooking up exact solutions this way, but it always introduces a source term, i.e., an additional complication to the original equation. Honestly speaking, I don't really like source terms because a scheme could lose its formal accuracy unless the source term discretization is carefully designed [34, 59, 115].