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Osher's Approximate Riemann Solver (1D Euler) (Nishikawa, Dec. 1998)

$$F_{j+1/2} = f(U_j) + \int_j^{j+1} A^-(U)\,dU = f(U_{j+1}) - \int_j^{j+1} A^+(U)\,dU$$

or

$$F_{j+1/2} = \frac{1}{2}\left[f(U_j) + f(U_{j+1})\right] - \frac{1}{2}\left[\int_j^{j+1} A^+(U)\,dU - \int_j^{j+1} A^-(U)\,dU\right]$$

or

$$F_{j+1/2} = \frac{1}{2}\left[f(U_L) + f(U_R)\right] - \frac{1}{2}\int_L^R |A(U)|\,dU$$

where $L = j$ and $R = j+1$.

We carry out the integral to obtain a practical formula for 1D Euler eqns.

$$\frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad \text{or} \quad \frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0$$

right eigenvectors $\vec{r}^{(1)}$, $\vec{r}^{(2)}$, $\vec{r}^{(3)}$

eigenvalues $\lambda^{(1)} = u - c$, $\lambda^{(2)} = u$, $\lambda^{(3)} = u + c$ $\quad (c^2 = \gamma P/\rho)$

Char. variables $V = [v^{(1)}, v^{(2)}, v^{(3)}]^T$

Cons. variables $U = [u^{(1)}, u^{(2)}, u^{(3)}]^T$

We have $df = A\,dU$ (But $df^{\pm} \neq A^{\pm} dU$), and $dU = R\,dV$

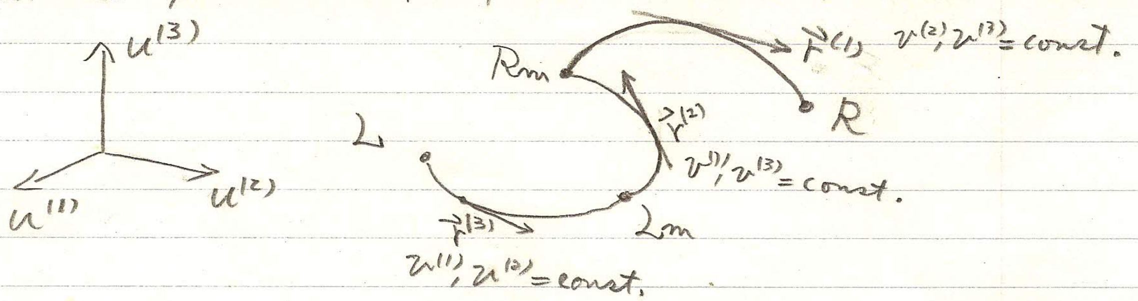
$$\to \quad df = A R\,dV \qquad (R = [\vec{r}^{(1)}, \vec{r}^{(2)}, \vec{r}^{(3)}])$$

$$\to \quad df = A\vec{r}^{(1)} dv^{(1)} + A\vec{r}^{(2)} dv^{(2)} + A\vec{r}^{(3)} dv^{(3)}$$

$$\to \quad \boxed{df = \lambda^{(1)}\vec{r}^{(1)} dv^{(1)} + \lambda^{(2)}\vec{r}^{(2)} dv^{(2)} + \lambda^{(3)}\vec{r}^{(3)} dv^{(3)}}$$

The integration path

The path is split over all simple wave solution as shown below.



So, we split the integral as follows

$$\int_L^R |A(U)|\,dU = \int_L^{Lm} |A(U)|\,dU + \int_{Lm}^{Rm} |A(U)|\,dU + \int_{Rm}^{R} |A(U)|\,dU$$

By definition of the path, (also note $dU = R dV = \vec{r}^{(1)} dv^{(1)} + \vec{r}^{(2)} dv^{(2)} + \vec{r}^{(3)} dv^{(3)}$)

$$\int_L^R |A(U)| dU = \int_L^{Lm} |A(U)| \vec{r}^{(3)} dv^{(3)} + \int_{Lm}^{Rm} |A(U)| \vec{r}^{(2)} dv^{(2)} + \int_{Rm}^{R} |A(U)| \vec{r}^{(1)} dv^{(1)}$$

$$= \int_L^{Lm} |\lambda^{(3)}| \vec{r}^{(3)} dv^{(3)} + \int_{Lm}^{Rm} |\lambda^{(2)}| \vec{r}^{(2)} dv^{(2)} + \int_{Rm}^{R} |\lambda^{(1)}| \vec{r}^{(1)} dv^{(1)}$$

$$\longrightarrow \boxed{\int_L^R |A(U)| dU = \int_L^{Lm} |\lambda^{(3)}| dU + \int_{Lm}^{Rm} |\lambda^{(2)}| dU + \int_{Rm}^{R} |\lambda^{(1)}| dU}$$

Also, we have along each path

$$\boxed{df = \lambda^{(k)} dU \quad k = 1, 2, 3}$$

We will use these to obtain the formula in terms of f. But before that, we find Lm and Rm.

Along each path (simple wave), we have the following relations for perfect gases.

$$\int_L^{Lm} : \; u_{Lm} - \frac{2}{\gamma-1} c_{Lm} = u_L - \frac{2}{\gamma-1} c_{Lm}, \quad P_{Lm}/\rho_{Lm}^{\gamma} = P_L/\rho_L^{\gamma}$$

$$\int_{Lm}^{Rm} : \; u_{Rm} = u_{Lm}, \quad P_{Rm} = P_{Lm}$$

$$\int_{Rm}^{R} : \; u_{Rm} + \frac{2}{\gamma-1} c_{Rm} = u_R + \frac{2}{\gamma-1} c_R, \quad P_{Rm}/\rho_{Rm}^{\gamma} = P_R/\rho_R^{\gamma}$$

Solving these 6 equations for 6 unknowns, we obtain

$$\boxed{\begin{aligned} P_{Lm} &= \left[\frac{\frac{\gamma-1}{2}[u_R - u_L] + c_R + c_L}{c_L\left[1 + \sqrt{\frac{\rho_L}{\rho_R}\left(\frac{P_R}{P_L}\right)^{1/\gamma}}\right]} \right]^{\frac{2\gamma}{\gamma-1}} P_L \\ \rho_{Lm} &= (P_{Lm}/P_L)^{1/\gamma} \rho_L, \quad u_{Lm} = u_L - \frac{2}{\gamma-1}(c_L - c_{Lm}) \\ P_{Rm} &= P_{Lm}, \quad u_{Rm} = u_{Lm}, \quad \rho_{Rm} = (P_{Rm}/P_R)^{1/\gamma} \rho_R \end{aligned}}$$

Now, we have everything that determines the states Lm and Rm, and we are ready for the integration.

Integration

We begin with the second one, $\int_{L_m}^{R_m} |\lambda^{(2)}| \, dU$

This is a linearly degenerate field, i.e. $\lambda^{(2)}$ is const. along the vector field $\vec{r}^{(2)}$, $\frac{d\lambda^{(2)}}{dU} \cdot \vec{r}^{(2)} = 0$. Therefore $\lambda^{(2)}$ is always of one sign.

$$\int_{L_m}^{R_m} |\lambda^{(2)}| \, dU = \begin{cases} \int_{L_m}^{R_m} \lambda^{(2)} dU = \int_{L_m}^{R_m} df & \text{for } \lambda_{Lm}^{(2)} \geq 0 \\ -\int_{L_m}^{R_m} \lambda^{(2)} dU = -\int_{L_m}^{R_m} df & \text{for } \lambda_{Lm}^{(2)} < 0 \end{cases}$$

(Note that $\lambda^{(2)} = u$, and $\Delta f = 0$ when $u = 0$.). Hence we write

$$\boxed{\int_{L_m}^{R_m} |\lambda^{(2)}| \, dU = \mathrm{sgn}(\lambda_{Lm}^{(2)}) \left[F(U_{Rm}) - F(U_{Lm}) \right]}$$

contribution from the entropy wave.

Next consider the first one, $\int_{L}^{L_m} |\lambda^{(3)}| \, dU$.

Observe

$$\frac{d\lambda^{(k)}}{dU} \cdot \vec{r}^{(k)} = 1 \quad (k=1,3) \text{ for normalized } \vec{r}^{(k)} \; (k=1,3)$$

$\rightarrow$ $\lambda^{(k)}$ is a monotone function along the integral curve.

$\rightarrow$ Let $\lambda \in [\lambda_1, \lambda_2]$, then

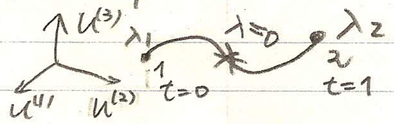
$$\begin{cases} \lambda > 0 & \lambda_1, \lambda_2 > 0 \\ \lambda < 0 & \lambda_1, \lambda_2 < 0 \\ \lambda = 0 \text{ once in } [\lambda_1, \lambda_2] & \lambda_1 \cdot \lambda_2 < 0 \end{cases}$$

(sonic point)

Hence, we need the state U^* at which $\lambda = 0$ to carry out the integral when $\lambda_L^{(3)} \cdot \lambda_{Lm}^{(3)} < 0$.

Since we know λ varies linearly, we parametrize the path by

$$\lambda = (1-t)\lambda_1 + t\lambda_2$$



then, $\lambda = 0$ at $\boxed{t = \frac{-\lambda_1}{\lambda_2 - \lambda_1}}$ and so $1 - t = \frac{\lambda_2}{\lambda_2 - \lambda_1}$.

Hence ($\lambda = u \pm c$), u^* and c^* (also vary linearly) are given by

$$\boxed{\begin{aligned} u^* &= \frac{\lambda_2}{\lambda_2-\lambda_1} u_1 - \frac{\lambda_1}{\lambda_2-\lambda_1} u_2 \\ c^* &= \frac{\lambda_2}{\lambda_2-\lambda_1} u_1 - \frac{\lambda_1}{\lambda_2-\lambda_1} u_2 \end{aligned}}$$

and

$$\boxed{\begin{aligned} p^* &= (c^*/c_1)^{\frac{2\gamma}{\gamma-1}} p_1 \\ \rho^* &= \gamma p^*/c^{*2} \end{aligned}}$$

We have U^*. Now back to the integral.

$$\int_L^{Lm} |\lambda^{(3)}| dU = \int_L^{Lm} \lambda^{(3)+} dU - \int_L^{Lm} \lambda^{(3)-} dU \qquad (\lambda^{(3)} = u+c)$$

We perform the integrals separately.

$$\int_L^{Lm} \lambda^{(3)+} dU = \begin{cases} f(U_{Lm}) - f(U_L) & \lambda_L^{(3)}, \lambda_{Lm}^{(3)} > 0 \\ 0 & \lambda_L^{(3)}, \lambda_{Lm}^{(3)} \leq 0 \\ f(U^*) - f(U_L) & \lambda_L^{(3)} > 0, \lambda_{Lm}^{(3)} < 0 \\ f(U_{Lm}) - f(U^*) & \lambda_L^{(3)} < 0, \lambda_{Lm}^{(3)} > 0 \end{cases}$$

$$\int_L^{Lm} \lambda^{(3)-} dU = \begin{cases} 0 & \lambda_L^{(3)}, \lambda_{Lm}^{(3)} > 0 \\ f(U_{Lm}) - f(U_L) & \lambda_L^{(3)}, \lambda_{Lm}^{(3)} < 0 \\ f(U_{Lm}) - f(U^*) & \lambda_L^{(3)} > 0, \lambda_{Lm}^{(3)} < 0 \\ f(U^*) - f(U_L) & \lambda_L^{(3)} < 0, \lambda_{Lm}^{(3)} > 0 \end{cases}$$

Put both together,

$$\boxed{\int_L^{Lm} |\lambda^{(3)}| dU = \begin{cases} f(U_{Lm}) - f(U_L) & \lambda_L^{(3)}, \lambda_{Lm}^{(3)} \geq 0 \\ f(U_L) - f(U_{Lm}) & \lambda_L^{(3)}, \lambda_{Lm}^{(3)} < 0 \\ -f(U_L) + 2f(U^*) - f(U_{Lm}) & \lambda_L^{(3)} > 0, \lambda_{Lm}^{(3)} < 0 \\ f(U_L) - 2f(U^*) + f(U_{Lm}) & \lambda_L^{(3)} < 0, \lambda_{Lm}^{(3)} > 0 \end{cases}}$$

In a similar way,

$$\boxed{\int_{Rm}^{R} |\lambda^{(1)}| dU = \begin{cases} f(U_R) - f(U_{Rm}) & \lambda^{(1)}_{Rm}, \lambda^{(1)}_R \geq 0 \\ f(U_{Rm}) - f(U_R) & \lambda^{(1)}_{Rm}, \lambda^{(1)}_R < 0 \\ -f(U_{Rm}) + 2f(U^*) - f(U_R) & \lambda^{(1)}_{Rm} > 0, \lambda^{(1)}_R < 0 \\ f(U_{Rm}) - 2f(U^*) + f(U_R) & \lambda^{(1)}_{Rm} < 0, \lambda^{(1)}_R > 0 \end{cases}}$$

Finally, we have everything necessary to compute

$$F_{j+1/2} = \frac{1}{2}[f(U_L) + f(U_R)] - \frac{1}{2}\left[\int_L^{Lm} |\lambda^{(3)}| dU + \int_{Lm}^{Rm} |\lambda^{(2)}| dU + \int_{Rm}^{R} |\lambda^{(1)}| dU\right].$$

But we can simplify the formulas a little further. Observe that for $\int_L^{Lm} |\lambda^{(3)}| dU$, we can write

$$\int_L^{Lm} |\lambda^{(3)}| dU = \begin{cases} \operatorname{sgn}(\lambda^{(3)}_L)[f(U_{Lm}) - f(U_L)] & \lambda^{(3)}_L \cdot \lambda^{(3)}_{Lm} \geq 0 \\ \operatorname{sgn}(\lambda^{(3)}_L)[-f(U_L) + 2f(U^*) - f(U_{Lm})] & \lambda^{(3)}_L \cdot \lambda^{(3)}_{Lm} < 0 \end{cases}$$

The second one can be written as

$$\underbrace{\operatorname{sgn}(\lambda^{(3)}_L)[f(U_{Lm}) - f(U_L)]} + 2\operatorname{sgn}(\lambda^{(3)}_L)[f(U^*) - f(U_{Lm})]$$

Hence, we can write

$$\int_L^{Lm} |\lambda^{(3)}| dU = \operatorname{sgn}(\lambda^{(3)}_L)[f(U_{Lm}) - f(U_L)] + \begin{cases} 0 & \lambda^{(3)}_L \cdot \lambda^{(3)}_{Lm} > 0 \\ 2\operatorname{sgn}(\lambda^{(3)}_L)[f(U^*) - f(U_{Lm})] & \lambda^{(3)}_L \cdot \lambda^{(3)}_{Lm} < 0 \end{cases}$$

The second term on the right can be considered as entropy correction.

Similarly, for $\int_{Rm}^{R} |\lambda^{(1)}| dU$,

$$\int_{Rm}^{R} |\lambda^{(1)}| dU = \operatorname{sgn}(\lambda^{(1)}_R)[f(U_R) - f(U_{Rm})] + \begin{cases} 0 & \lambda^{(1)}_R \cdot \lambda^{(1)}_{Rm} > 0 \\ 2\operatorname{sgn}(\lambda^{(1)}_R)[f(U_{Rm}) - f(U^*)] & \lambda^{(1)}_R \cdot \lambda^{(1)}_{Rm} < 0 \end{cases}$$

Finally, we obtain

$$\boxed{\begin{aligned}
&F_{j+1/2} = \frac{1}{2}\left[f(U_L)+f(U_R)\right] - \frac{1}{2}\left[F + F_{3\text{-sonic}} + F_{1\text{-sonic}}\right] \\
&\text{where} \\
&F = \operatorname{sgn}(\lambda_L^{(3)})\left[f(U_{Lm})-f(U_L)\right] + \operatorname{sgn}(\lambda_{Lm}^{(2)})\left[f(U_{Rm})-f(U_{Lm})\right] + \operatorname{sgn}(\lambda_R^{(1)})\left[f(U_R)-f(U_{Rm})\right] \\
&F_{3\text{-sonic}} = \begin{cases} 0 & \lambda_L^{(3)}\cdot\lambda_{Lm}^{(3)} > 0 \\ 2\operatorname{sgn}(\lambda_L^{(3)})\left[f(U^*)-f(U_{Lm})\right] & \lambda_L^{(3)}\cdot\lambda_{Lm}^{(3)} < 0 \end{cases} \\
&F_{1\text{-sonic}} = \begin{cases} 0 & \lambda_R^{(1)}\cdot\lambda_{Rm}^{(1)} > 0 \\ 2\operatorname{sgn}(\lambda_R^{(1)})\left[f(U_{Rm})-f(U^*)\right] & \lambda_R^{(1)}\cdot\lambda_{Rm}^{(1)} < 0 \end{cases}
\end{aligned}}$$