Revised version of the AIAA2014-2091 presentation.

First, Second, and Third Order Hyperbolic Navier-Stokes Solver

Hiroaki Nishikawa

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NIA CFD Seminar

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Myth

CFD Myth #1

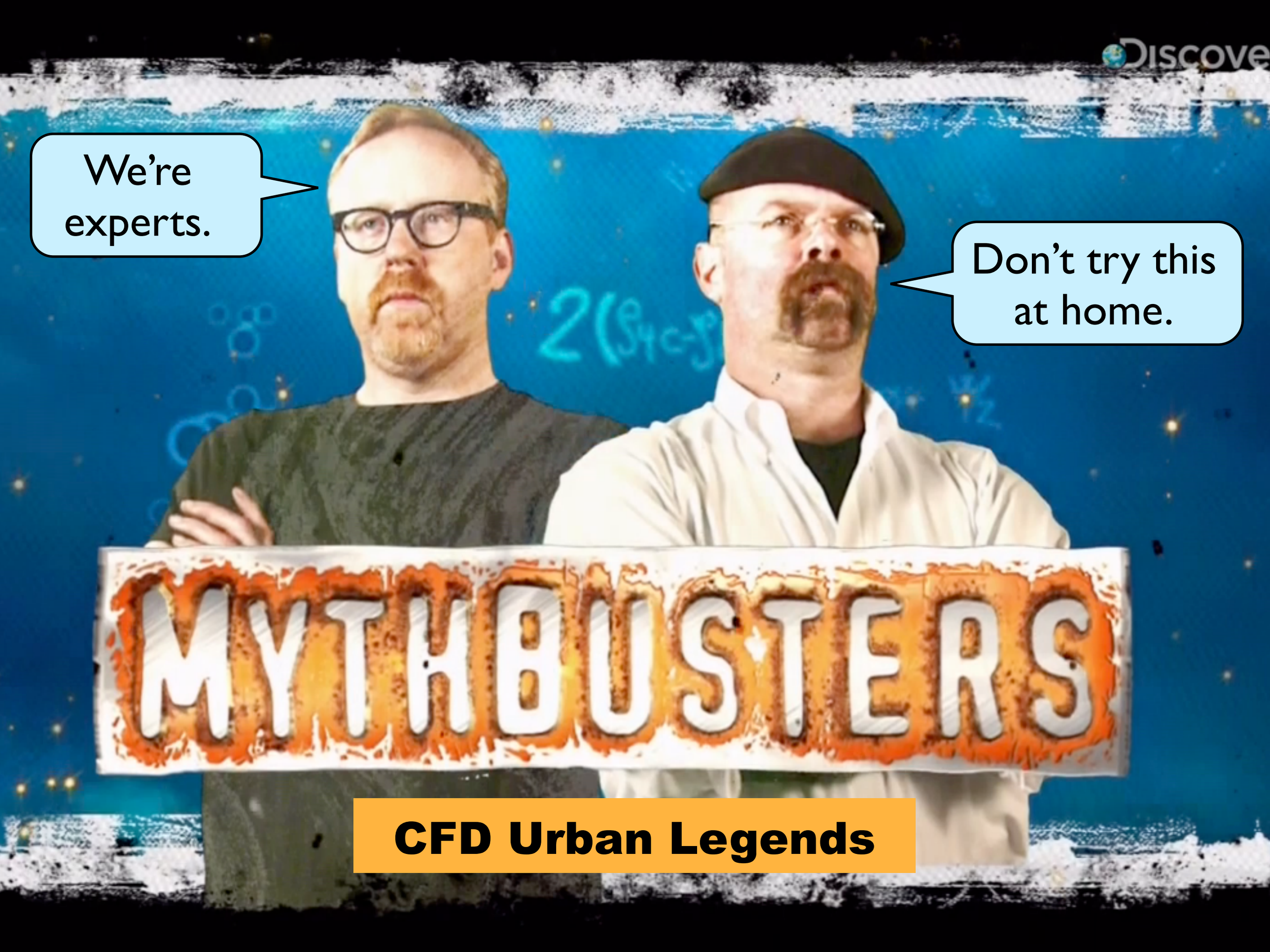
Nice and smooth viscous grid:

“ Boundary-layer grid must be smooth and orthogonal. “

CFD Myth #2

Impossible High-Order Scheme:

“ There exists a 3rd-order scheme that is less expensive on a given grid than a 2nd-order FV scheme widely used today. “

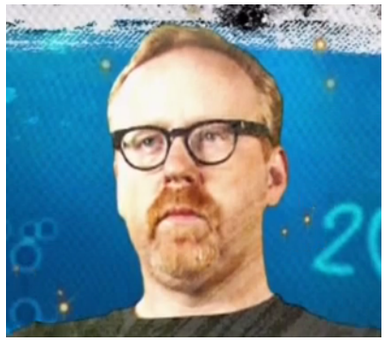


CFD Urban Legends

Impossible high-order scheme...

Jamie:

"Sounds impossible."



Adam:

"Nothing is impossible.
It only takes a crazy idea to make it possible."

Jamie:

"OK, let's search in Google."

https www.google.com/#q=fast+convergence%2C+Navier-Stokes%2C+accurate%2C+radical — fast convergence, Navier-Stokes, accurate, radical - Google Se

fast convergence, Navier-Stokes, accurate, radical

+Hiro

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About 187,000 results (0.34 seconds)

[PDF] New-Generation Hyperbolic **Navier-Stokes** Schemes: O(...
www.hiroakinishikawa.com/My.../nishikawa_AIAA-2011-3043.pdf
by H Nishikawa - 2011 - Cited by 5 - Related articles
be **radically** different from those currently used. ... **Navier-Stokes** code, and demonstrate the O(1/h) speed-up and **accurate** viscous/heat fluxes on irregular ..... stresses and the heat fluxes comes with the O(1/h) **faster** iterative **convergence**.
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Papers and Publications by Hiro(aki) Nishikawa
www.hiroakinishikawa.com/papers.html
H. Nishikawa, Two Ways to Extend Diffusion Schemes to **Navier-Stokes** ... **Fast** grid-independent **convergence** is demonstrated for mixed-element grids ... report on the development of **radically** new **Navier-Stokes** schemes: uniform **accuracy**, ...

[PDF] Solution methods for the Incompressible **Navier-Stokes** ...
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Baker, G.: Galerkin approximations for the **Navier-Stokes** equations. ... Connors, J.M.: Convergence analysis and computational testing of the finite element ..... In this article we manifest a **radically** different counting procedure first ... this quantum approach achieves much **faster convergence** rate than the classical approach.

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AIAA 2011-3043

20th AIAA Computational Fluid Dynamics Conference, 27 - 30 June, Honolulu, Hawaii, 2011

New-Generation Hyperbolic Navier-Stokes Schemes: $O(1/h)$ Speed-Up and Accurate Viscous/Heat Fluxes

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In this paper, we introduce a first-order hyperbolic system model for viscous flows, and propose a unique way of computing steady viscous flows: integrate the hyperbolic Navier-Stokes system in time toward the steady state. We construct an upwind finite-volume scheme for the hyperbolic Navier-Stokes system and demonstrate remarkable advantages of the resulting Navier-Stokes code: $O(1/h)$ speed-up over traditional Navier-Stokes codes, where h is the mesh spacing, and the capability of simultaneously computing the viscous stresses and the heat fluxes to the same order of accuracy as that of the main variables on irregular grids. The paper concludes with discussions on the future developments and the potential impact on the future algorithm development for computational fluid dynamics.

1. Introduction

In this paper, we extend the first-order hyperbolic system method developed for model equations in a series of papers [1,2] to the Navier-Stokes equations. We thereby propose a non-traditional way of computing viscous flows: integrate an equivalent first-order hyperbolic system in time toward the steady state. The first-order system is deliberately designed such that it reduces to the original Navier-Stokes equations in the steady state; its viscous part is a *hyperbolic* system on its own. Navier-Stokes codes arising from the proposed method will be radically different from those currently used.

Hyperbolic Method Since JCP2007

Traditional NS System

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p - \tau)}{\partial x} = 0,$$

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho u H - \tau u + q)}{\partial x} = 0,$$

$$\tau = \mu_v \frac{\partial u}{\partial x},$$

$$q = -\frac{\mu_h}{\gamma(\gamma-1)} \frac{\partial T}{\partial x}.$$

$$\left(\mu_v = \frac{4}{3}\mu, \quad \mu_h = \frac{\gamma\mu}{Pr}.\right)$$

Hyperbolic NS System

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p - \tau)}{\partial x} = 0,$$

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho u H - \tau u + q)}{\partial x} = 0,$$

$$\frac{\partial \tau}{\partial t} - \frac{1}{T_v}\left(\mu_v \frac{\partial u}{\partial x} - \tau\right) = 0,$$

$$\frac{\partial q}{\partial t} - \frac{1}{T_h}\left(-\frac{\mu_h}{\gamma(\gamma-1)} \frac{\partial T}{\partial x} - q\right) = 0,$$

Nishikawa, AIAA 2011-3043

Free parameters: $T_v = L^2/\nu_v, \quad T_h = L^2/\nu_h$

Two systems are equivalent in the steady state.

See NASA-TM2014 and CF2014 for **unsteady** schemes.
See AIAA 2014-2091 for hyperbolic **incompressible** NS system.

Nishikawa, AIAA 2011-3043

Preconditioned Conservative System

$$\mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}$$

Inviscid

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & T_v/\mu_v & 0 \\ 0 & 0 & 0 & 0 & T_h/\mu_h \end{bmatrix}, \mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \\ \tau \\ q \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p - \tau \\ \rho u H - \tau u + q \\ -u \\ \dfrac{a^2}{\gamma(\gamma-1)} \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau/\mu_v \\ q/\mu_h \end{bmatrix}$$

Inviscid and Viscous Jacobians:

$$\mathbf{PA} = \mathbf{PA^i} + \mathbf{PA^v}, \quad \mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}.$$

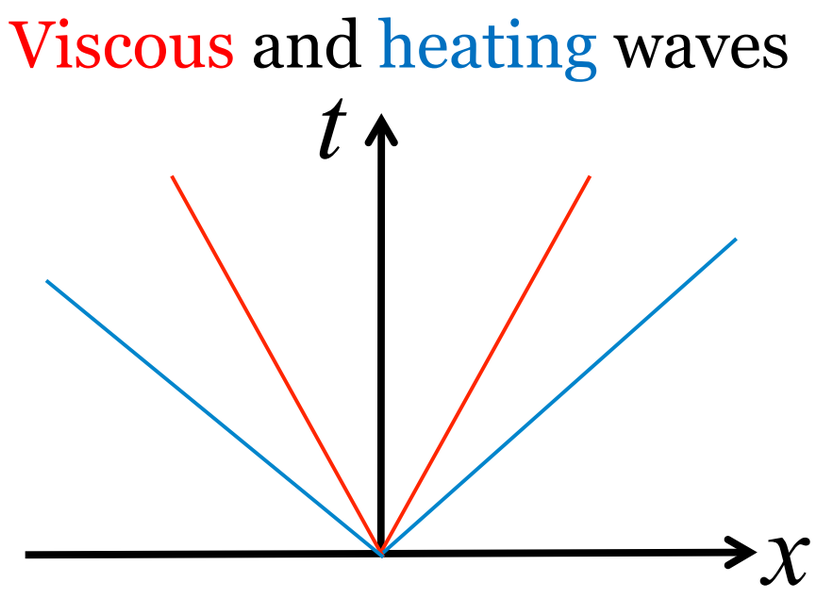
Eigen-structure of each Jacobian is fully analyzable.

Nishikawa, AIAA 2011-3043

Viscous Terms are Hyperbolic

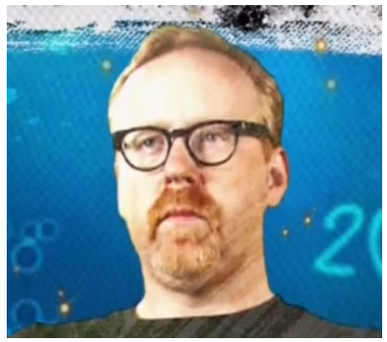
Viscous Jacobian has real eigenvalues:

$$\lambda^v = \pm\sqrt{\frac{\nu_v}{T_v}}, \quad \pm\sqrt{\frac{\nu_h}{T_h}}, \quad 0$$



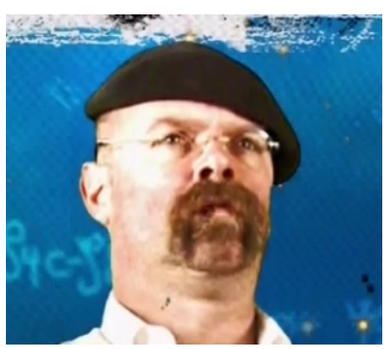
Navier-Stokes Equations = Hyperbolic Inviscid + Hyperbolic Viscous

All we need are methods for hyperbolic systems, e.g., upwind.



Adam:

"Upwind viscous scheme is crazy, and I love it!"



Jamie:

"If true, there's a good chance to confirm the myths."

JCP2007, 2010, 2012, 2013,2014, AIAA2011,
NASA-TM2014, CF2014

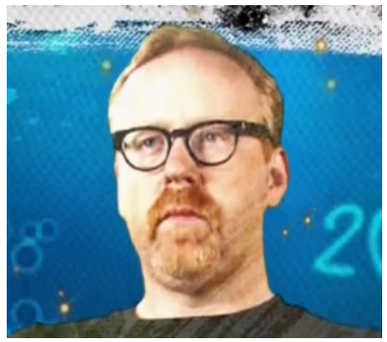
Advantage of Hyperbolic Method

1. Discretization made simple and straightforward
- Schemes and techniques developed for advection can be directly applied to viscous terms - *multi-dimensional upwind, high-order, etc.*
- 1st-order viscous schemes (robustness, consistent Jacobian, P0 DG)

2. O(1/h) acceleration in convergence (no second derivatives)
- O(h) time step for explicit schemes for steady problems
- O(1/h) condition number for linear system in implicit schemes

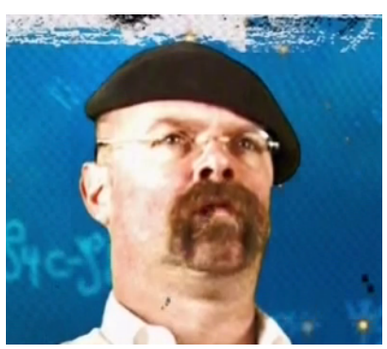
3. Higher-order derivatives (viscous/heat fluxes)
- Same order of accuracy for solution and derivatives on irregular grids
- Highly smooth derivatives on skewed irregular grids

If you have a good inviscid scheme,
you have a <u>very</u> good viscous scheme.



Adam:

“ Sounds great!
Let’s try it out. ”



Jamie:

“ All right.
Let’s discretize it by a FV scheme”

Edge-Based Finite-Volume Method

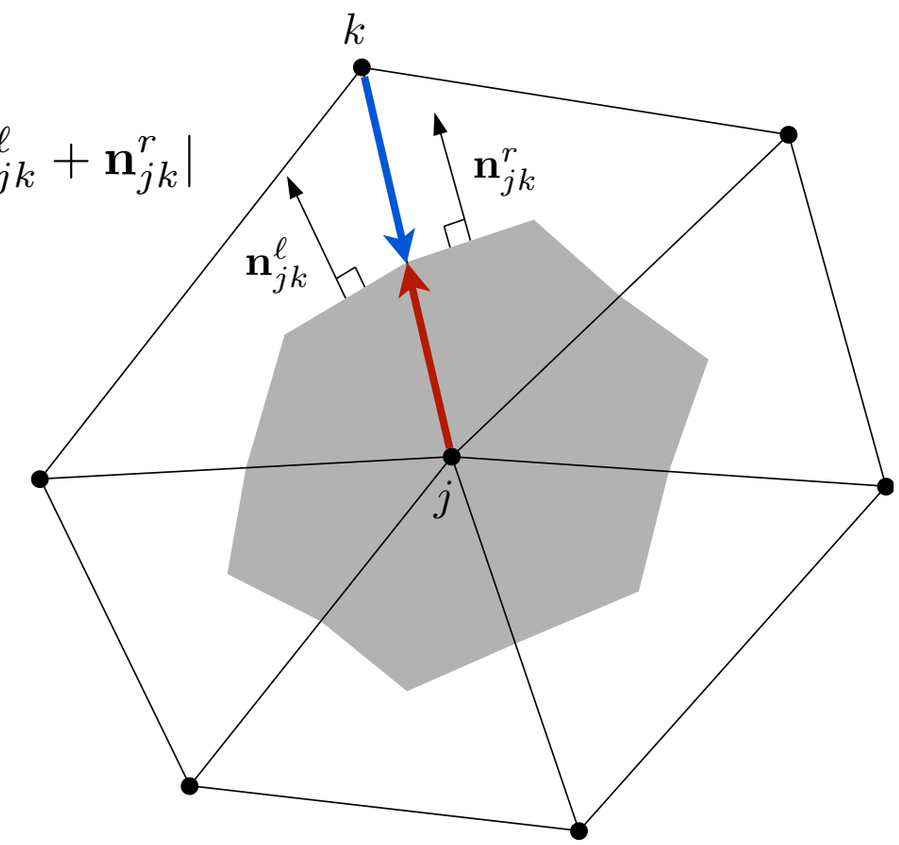
NASA's FUN3D; Software Cradle's SC/Tetra; DLR Tau code, etc.

$$\mathbf{P}^{-1}\partial_t \mathbf{U} + \partial_x \mathbf{F} + \partial_y \mathbf{G} = \mathbf{S}$$

Edge-based finite-volume scheme:

$$V_j \frac{d\mathbf{U}_j}{dt} = -\mathbf{P}_j \left(\sum_{k \in \{k_j\}} \Phi_{jk} A_{jk} - \mathbf{S}_j V_j \right)$$

$$A_{jk} = |\mathbf{n}^{\ell}_{jk} + \mathbf{n}^{r}_{jk}|$$



with the upwind flux at edge midpoint:

$$\Phi_{jk} = \frac{1}{2}(\mathbf{H}_{nL} + \mathbf{H}_{nR}) - \frac{1}{2}|\mathbf{A}_n|(\mathbf{U}_R - \mathbf{U}_L)$$

Accuracy determined by left/right states:

- 1st-order with nodal values
- 2nd-order with linear extrapolation, linear LSQ

1st/2nd-order for both inviscid and viscous terms.

Numerical Flux for Navier-Stokes

Sum of upwind inviscid and upwind viscous fluxes:

$$\mathbf{\Phi}_{jk} = \mathbf{\Phi}^{i}_{jk} + \mathbf{\Phi}^{v}_{jk}$$

Upwind Inviscid (Roe)

$$\mathbf{\Phi}^{i}_{jk} = \frac{1}{2}\left[\mathbf{H}^{i}_{R} + \mathbf{H}^{i}_{L}\right] - \left|\mathbf{A}^{i}_{n}\right|\left(\mathbf{U}_{R} - \mathbf{U}_{L}\right)$$

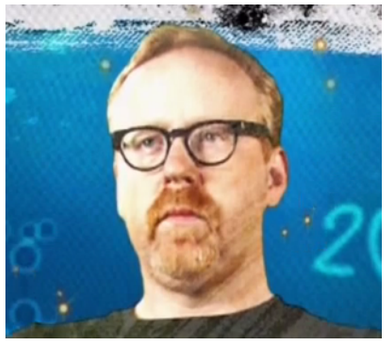
Upwind Viscous

$$\mathbf{\Phi}^{v}_{jk} = \frac{1}{2}\left[\mathbf{H}^{v}_{R} + \mathbf{H}^{v}_{L}\right] - \mathbf{P}^{-1}\left|\mathbf{P}\mathbf{A}^{v}_{n}\right|\left(\mathbf{U}_{R} - \mathbf{U}_{L}\right)$$

Standard Construction
in local-preconditioning technique

Jamie:

“ Looks radical.
But it’s not 3rd-order accurate... ”



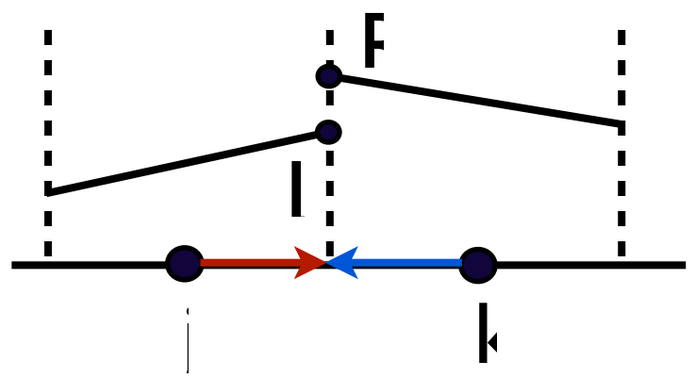
Adam:

“ Then, we make it 3rd-order! ”

Katz and Sankaran JCP2011

Third-Order EB FV Scheme for Hyperbolic Systems

For triangular/tetrahedral grids only.



1. 2nd-order gradients at nodes (e.g., LSQ quadratic fit).

2. Extrapolate flux/solution to the midpoint.

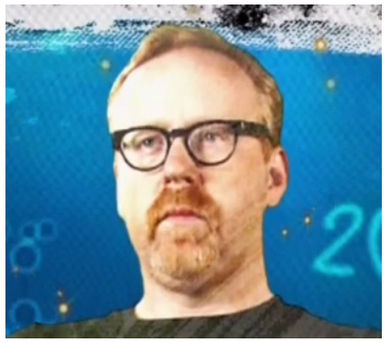
$$\mathbf{H}_L = \mathbf{H}_j + \frac{1}{2}\nabla\mathbf{H}_j \cdot \Delta\mathbf{l}_{jk}, \quad \mathbf{H}_R = \mathbf{H}_k - \frac{1}{2}\nabla\mathbf{H}_k \cdot \Delta\mathbf{l}_{jk}$$

Non-trivial for viscous and source terms (See, e.g., Nishikawa, JCP2014),

Trivial for hyperbolic NS system, but there's a source term.

Jamie:

“ What can we do with the source term? ”



Adam:

“ Stay crazy, and make it hyperbolic! ”

Divergence Form of Source

Nishikawa JCP2012

Write the source as a hyperbolic system:

$$\mathbf{S} \rightarrow \partial_x \mathbf{F}^s + \partial_y \mathbf{G}^s$$

$$\mathbf{F}^s = (x - x_j)\mathbf{S} - \frac{1}{2}(x - x_j)^2 \partial_x \mathbf{S} + \frac{1}{6}(x - x_j)^3 \partial_{xx} \mathbf{S}, \quad \mathbf{G}^s = 0$$

Hyperbolic NS system becomes a single conservation law:

$$\mathbf{P}^{-1}\partial_t \mathbf{U} + \partial_x \left(\mathbf{F} - \mathbf{F}^s\right) + \partial_y \left(\mathbf{G} - \mathbf{G}^s\right) = 0$$

Apply the same 3rd-order scheme to inviscid, viscous, and source terms.

$$V_j \frac{d\mathbf{U}_j}{dt} = -\mathbf{P}_j \left(\sum_{k \in \{k_j\}} \Phi'_{jk} A_{jk} \right)$$

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Numerical Flux for Navier-Stokes

Sum of upwind inviscid, upwind viscous, and upwind source fluxes:

$$\mathbf{\Phi}'_{jk} = \mathbf{\Phi}^i_{jk} + \mathbf{\Phi}^v_{jk} + \mathbf{\Phi}^s_{jk}$$

Upwind Inviscid (Roe)

$$\mathbf{\Phi}^i_{jk} = \frac{1}{2}\left[\mathbf{H}^i_R + \mathbf{H}^i_L\right] - \left|\mathbf{A}^i_n\right|\left(\mathbf{U}_R - \mathbf{U}_L\right)$$

Upwind Viscous

$$\mathbf{\Phi}^v_{jk} = \frac{1}{2}\left[\mathbf{H}^v_R + \mathbf{H}^v_L\right] - \mathbf{P}^{-1}\left|\mathbf{P}\mathbf{A}^v_n\right|\left(\mathbf{U}_R - \mathbf{U}_L\right)$$

Upwind Source (to replace the point source integration)

$$\mathbf{\Phi}^s_{jk} = \frac{1}{2}\left[\mathbf{H}^s_R + \mathbf{H}^s_L\right] - \mathbf{P}^{-1}\left|\mathbf{P}\mathbf{A}^s_n\right|\left(\mathbf{U}_R - \mathbf{U}_L\right)$$

Implicit Solver

Residual at node j (time derivative dropped) :

1st/2nd-order

$$0 = -\mathbf{P}_j \sum_{k \in \{k_j\}} \mathbf{\Phi}_{jk} A_{jk} + \mathbf{P}_j \mathbf{S}_j V_j$$

3rd-order

$$0 = -\mathbf{P}_j \sum_{k \in \{k_j\}} \mathbf{\Phi}'_{jk} A_{jk}$$

NOTE: Steady residual is consistent with the **original NS equations** because the time derivative does not exist any more.

System of residual equations : $0 = \mathbf{Res}(\mathbf{U}_h)$

Update : $$\mathbf{U}_h^{n+1} = \mathbf{U}_h^n + \Delta \mathbf{U}_h$$

$$\frac{\partial \mathbf{Res}}{\partial \mathbf{U}_h} \Delta \mathbf{U}_h = -\mathbf{Res}(\mathbf{U}_h^n)$$

- Jacobian exact for 1st-order scheme (Newton's method for 1st-order scheme)
- GS relaxation to reduce the linear residual by one order
- Run 1st-order to get initial solution for 2nd/3rd-order

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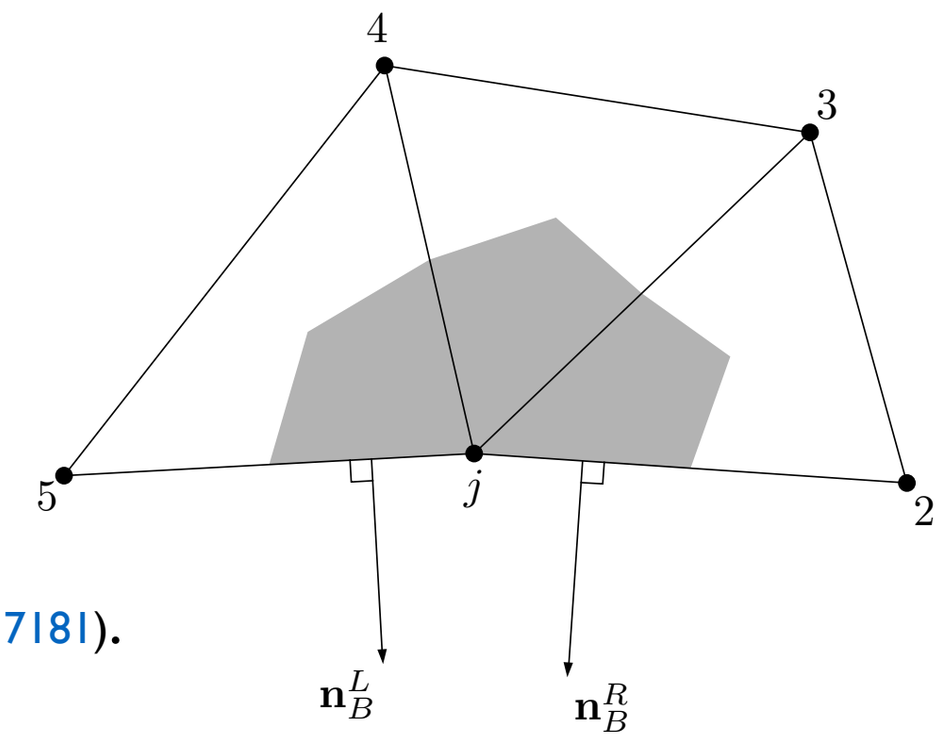
Boundary Conditions

Weak Condition:

$$\left(\frac{5}{6}\mathbf{\Phi}_j + \frac{1}{6}\mathbf{\Phi}_5\right)\cdot\mathbf{n}_B^L + \left(\frac{5}{6}\mathbf{\Phi}_j + \frac{1}{6}\mathbf{\Phi}_2\right)\cdot\mathbf{n}_B^R$$



- Specify condition as the right state (NASA-TM–2011-217181).

Strong Condition:

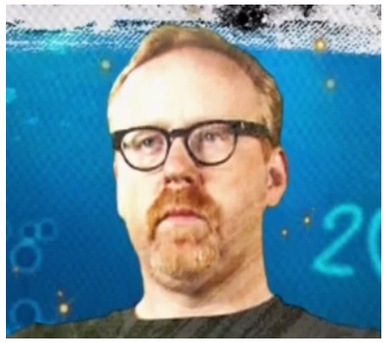
No slip condition

$$(u, v)_j = 0$$

Adiabatic condition (Neumann->Dirichlet)

$$(q_x, q_y)_j \cdot \mathbf{n}_j = 0$$

The viscous stresses are predicted by the scheme.



Adam:

"Hey, it slows down on stretched mesh!"

Jamie:

"Failure is always an option in Mythbusters. Let's perform some analysis. "

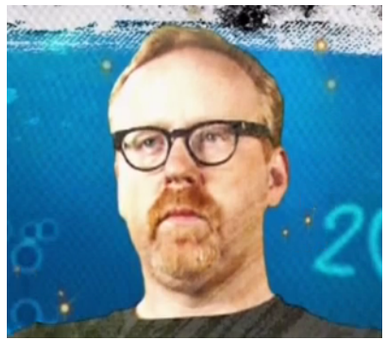
High-Aspect Ratio Grids

Analysis indicates that the hyperbolic scheme slows down on high aspect ratio grids, but it is resolved by the modified length scale:

$$L = \frac{1}{2\pi \mathcal{AR}}$$

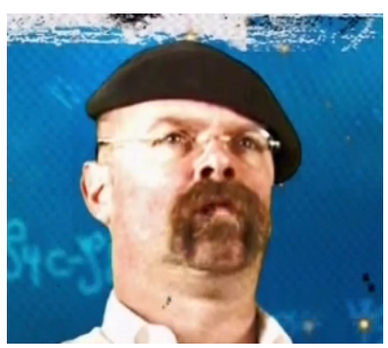
where AR is the local cell aspect ratio.

Details will be reported in a separate paper.



Adam:

“ It runs fine now.
But not very accurate on boundaries.....”



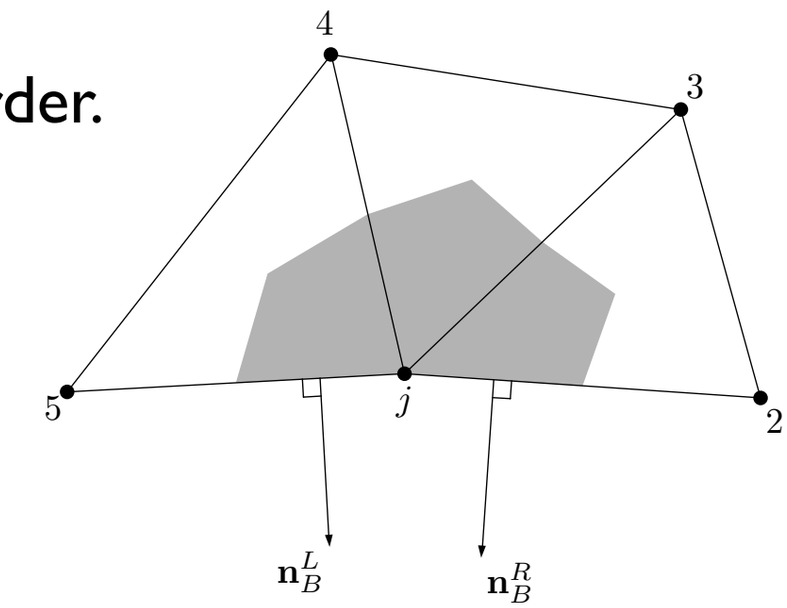
Jamie:

“ 2nd-order boundary quadrature doesn’t work for 3rd-order scheme.
We need to derive a new formula.”

Boundary Quadrature

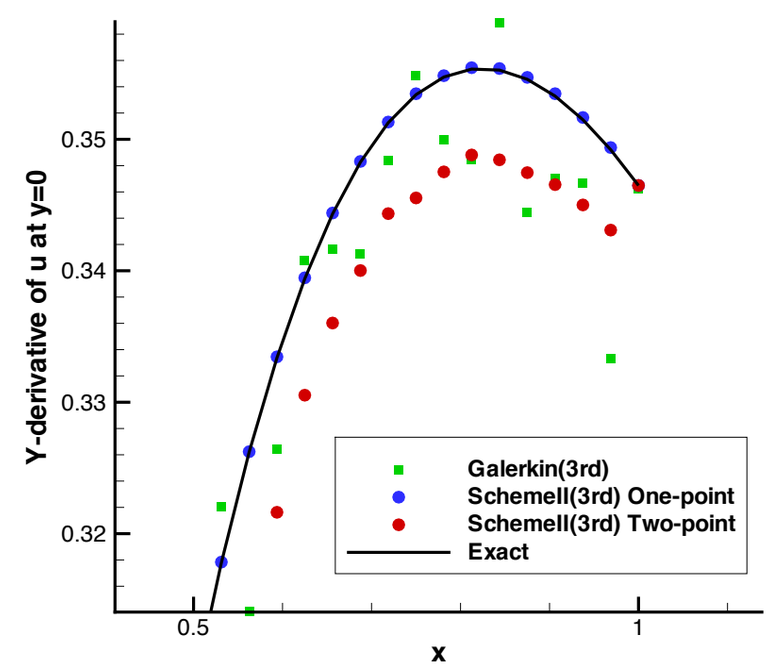
2nd-order boundary quadrature doesn't work for 3rd-order.

$$\cancel{\left(\frac{5}{6}\mathbf{\Phi}_j + \frac{1}{6}\mathbf{\Phi}_5\right) \cdot \mathbf{n}_B^L + \left(\frac{5}{6}\mathbf{\Phi}_j + \frac{1}{6}\mathbf{\Phi}_2\right) \cdot \mathbf{n}_B^R}$$

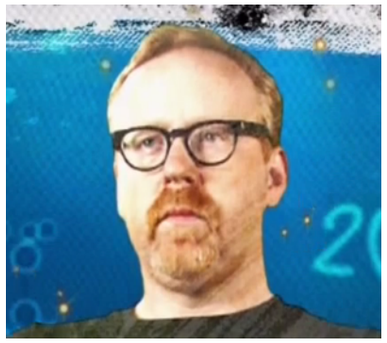


Derived a 3rd-order boundary quadrature formula:

$$\left(\frac{2}{3}\mathbf{\Phi}_{jb} + \frac{1}{3}\mathbf{\Phi}_{j5}\right) \cdot \mathbf{n}_B^L + \left(\frac{2}{3}\mathbf{\Phi}_{jb} + \frac{1}{3}\mathbf{\Phi}_{j2}\right) \cdot \mathbf{n}_B^R$$



A paper currently in review.



Adam:

"OK, I think it's working now!"

Jamie:

"Let's verify the accuracy first."

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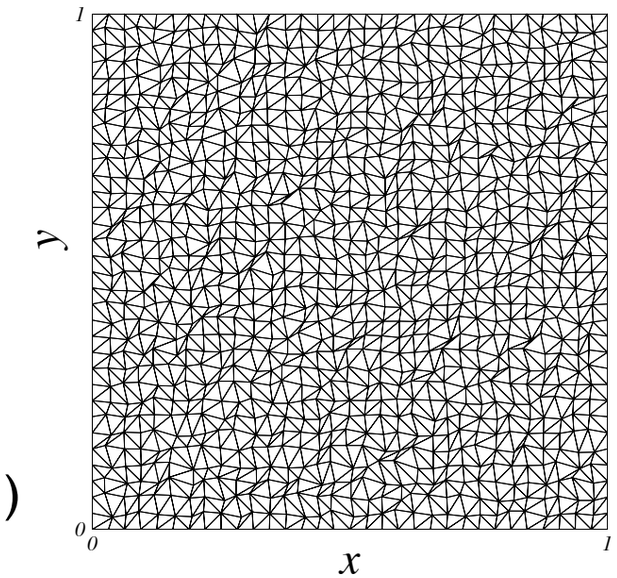
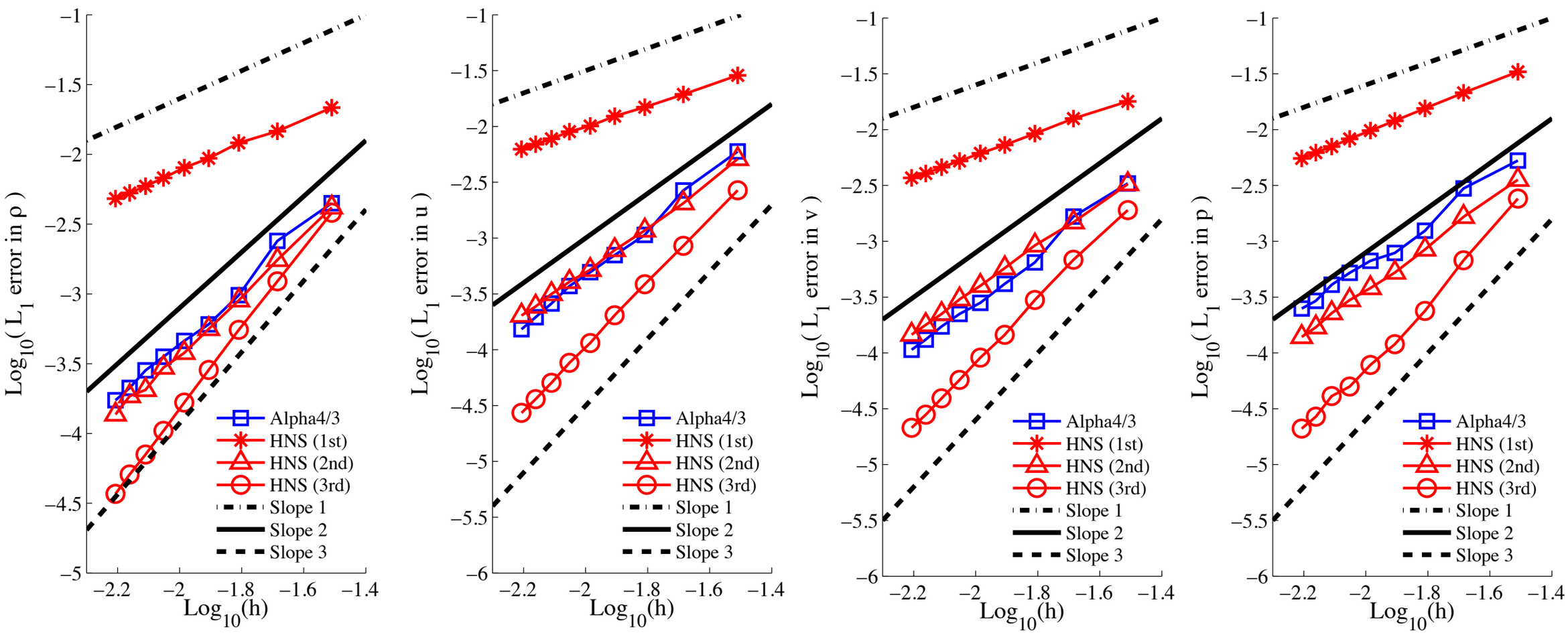
Accuracy Verification



Manufactured solution in a square domain, irregular triangular grids.
Alpha4/3 is a conventional viscous scheme (Nishikawa, AIAA2010-5093, C&F2011)

Mach 0.3, Re=100, Pr=3/4



1st, 2nd, and 3rd-order accuracy verified.

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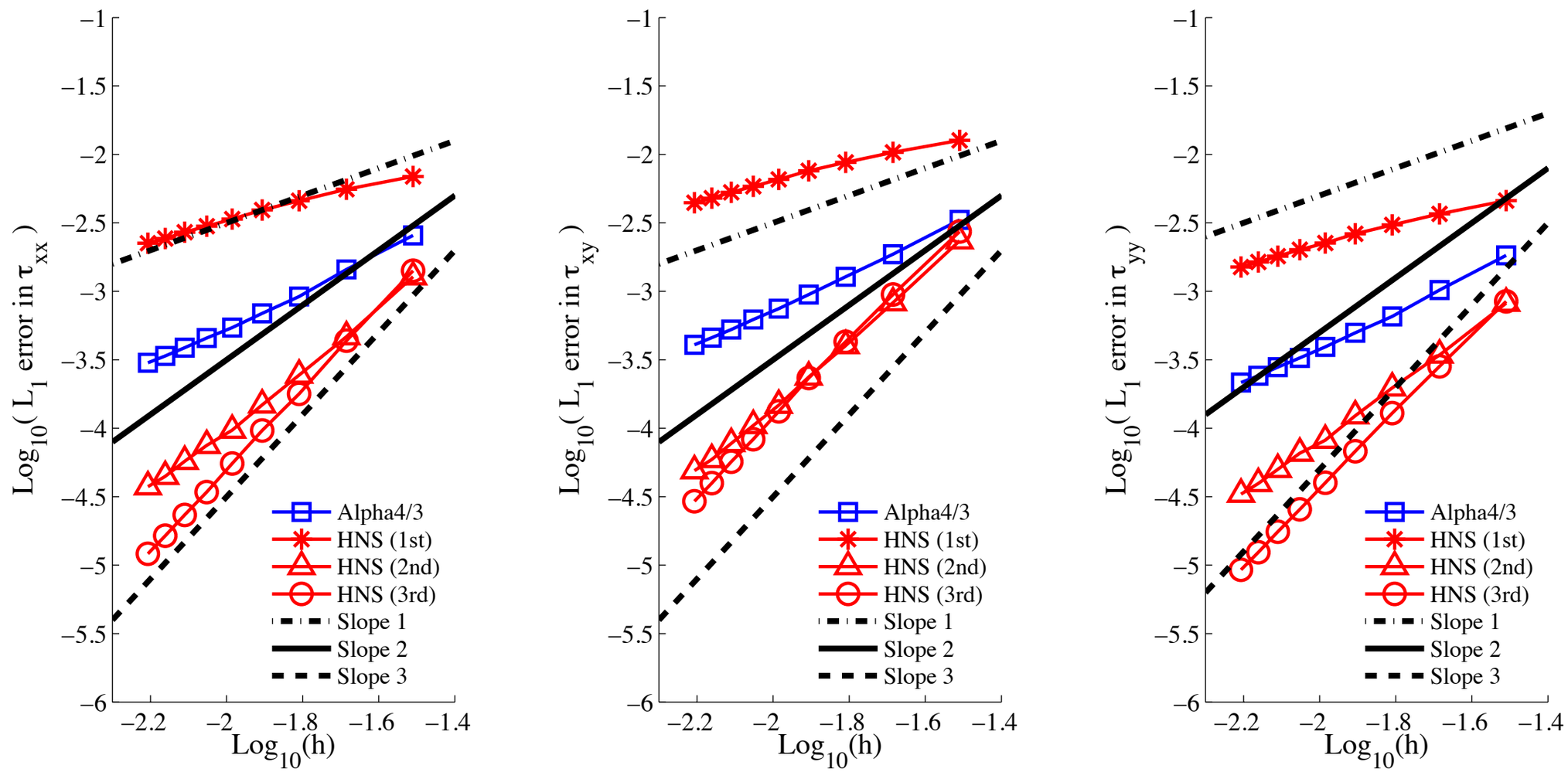
Accuracy Verification

Viscous Stresses



1st, 2nd, and 3rd-order accuracy verified.
Conventional scheme gives 1st-order accuracy.

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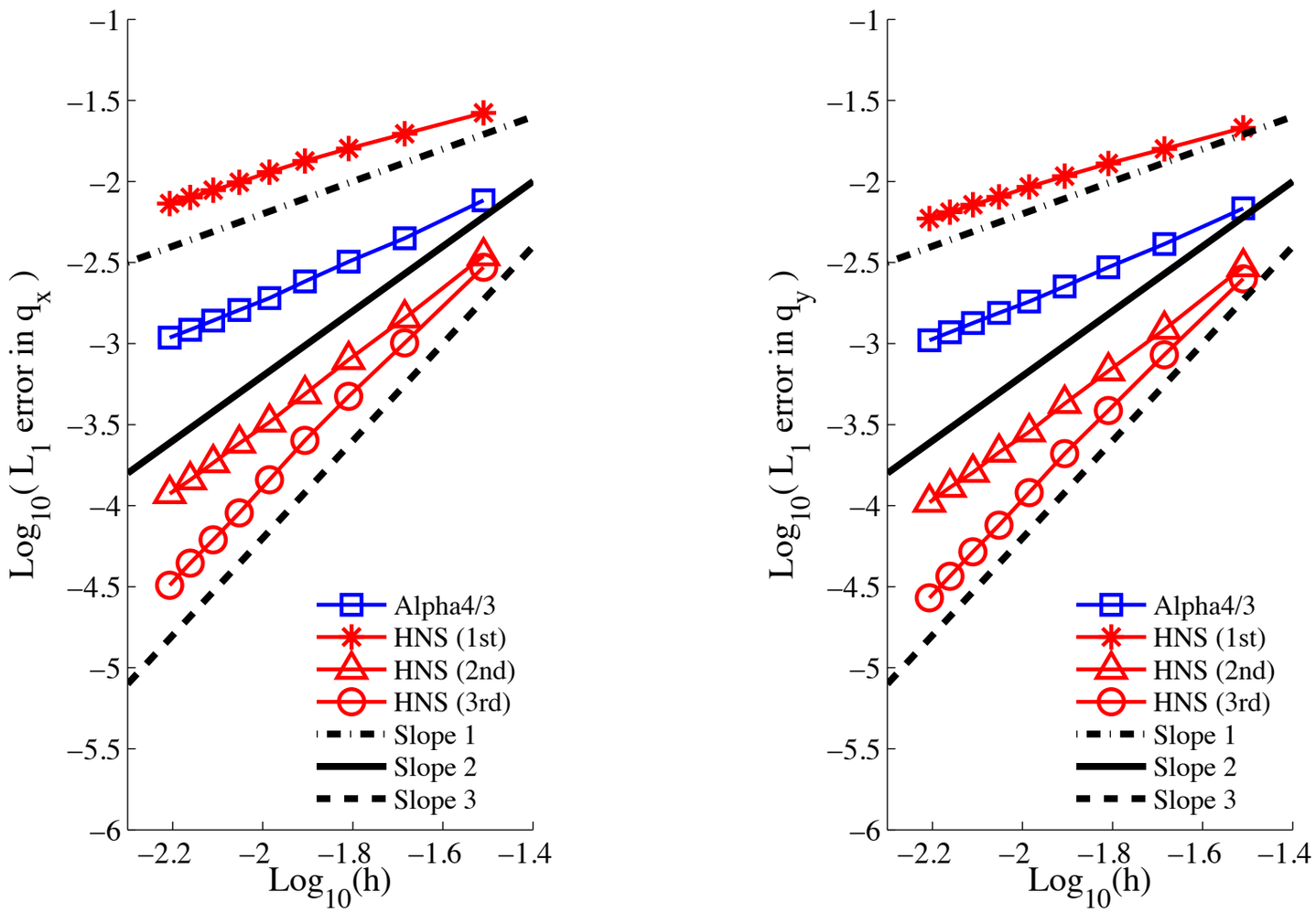
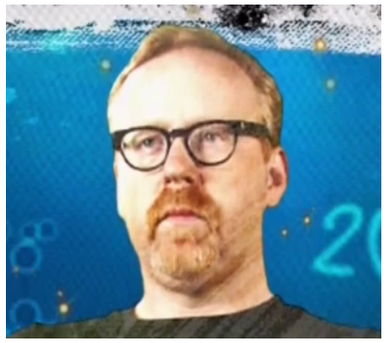
Accuracy Verification

Heat Fluxes

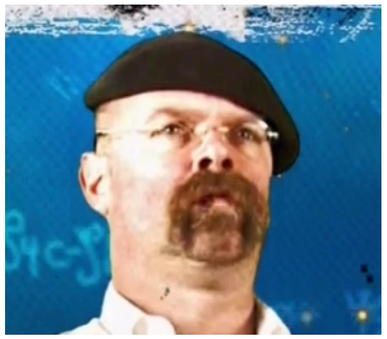


1st, 2nd, and 3rd-order accuracy verified.
Conventional scheme gives 1st-order accuracy.



Adam:

“ How about velocity gradients?
They are useful for many purposes. ”



Jamie:

“ That’s actually a difficult problem...”

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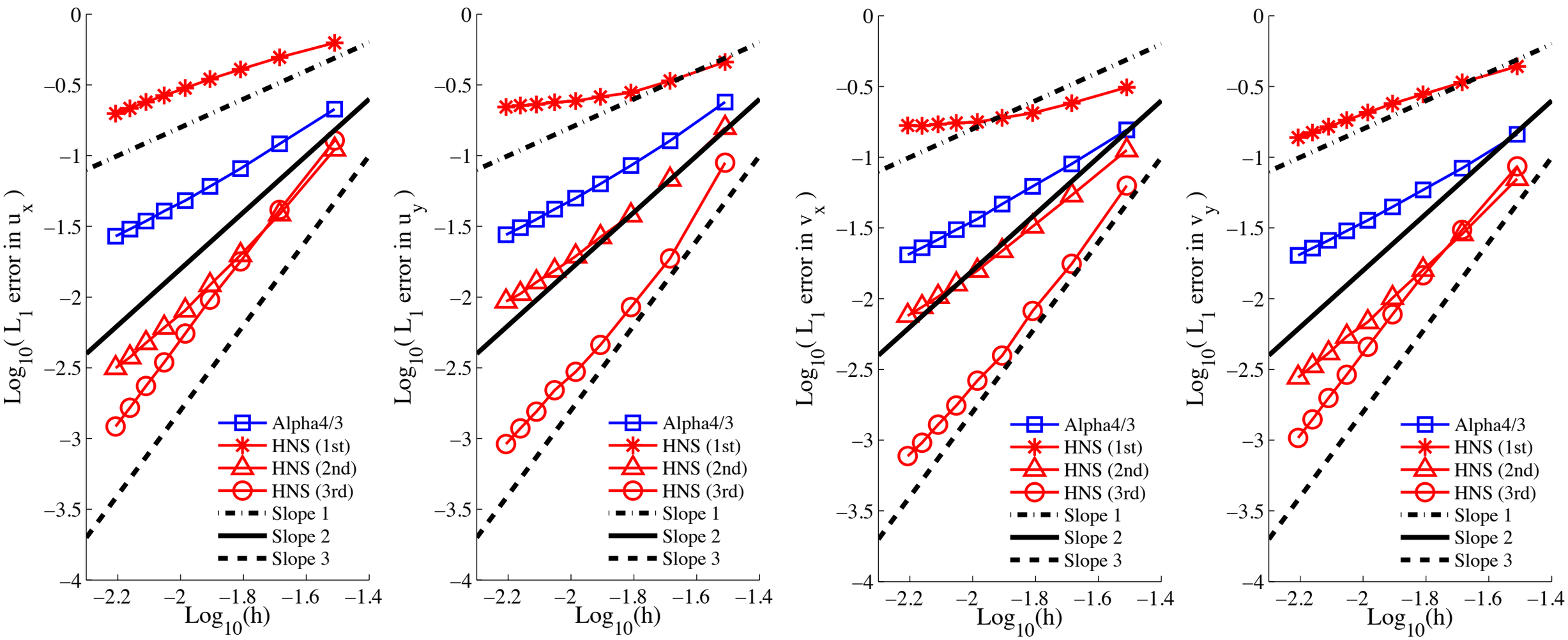
Accuracy Verification

Velocity Gradients

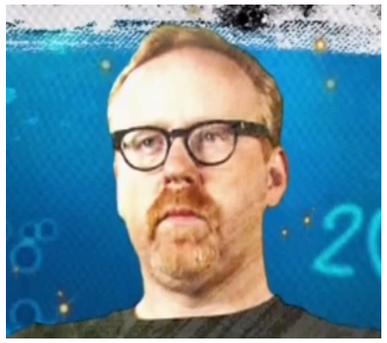
$$\frac{\tau_{xx}}{\mu_v} = \partial_x u - \frac{1}{2}\partial_y v, \quad \frac{\tau_{xy}}{\mu_v} = \frac{3}{4}(\partial_y u + \partial_x v), \quad \frac{\tau_{yy}}{\mu_v} = \partial_y v - \frac{1}{2}\partial_x u$$



1st/2nd/3rd-order for ux and vy.

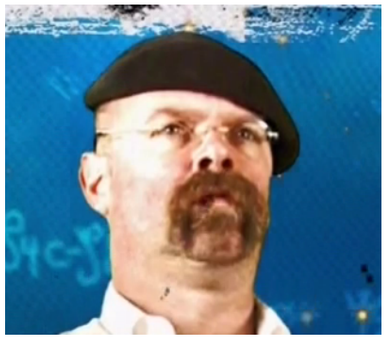
Still very accurate, but one order lower for uy and vx.

In 3D, all components will be one order lower...



Adam:

“ OK, let’s try some realistic problem. ”



Jamie:

“ How about a flow over a bump? ”

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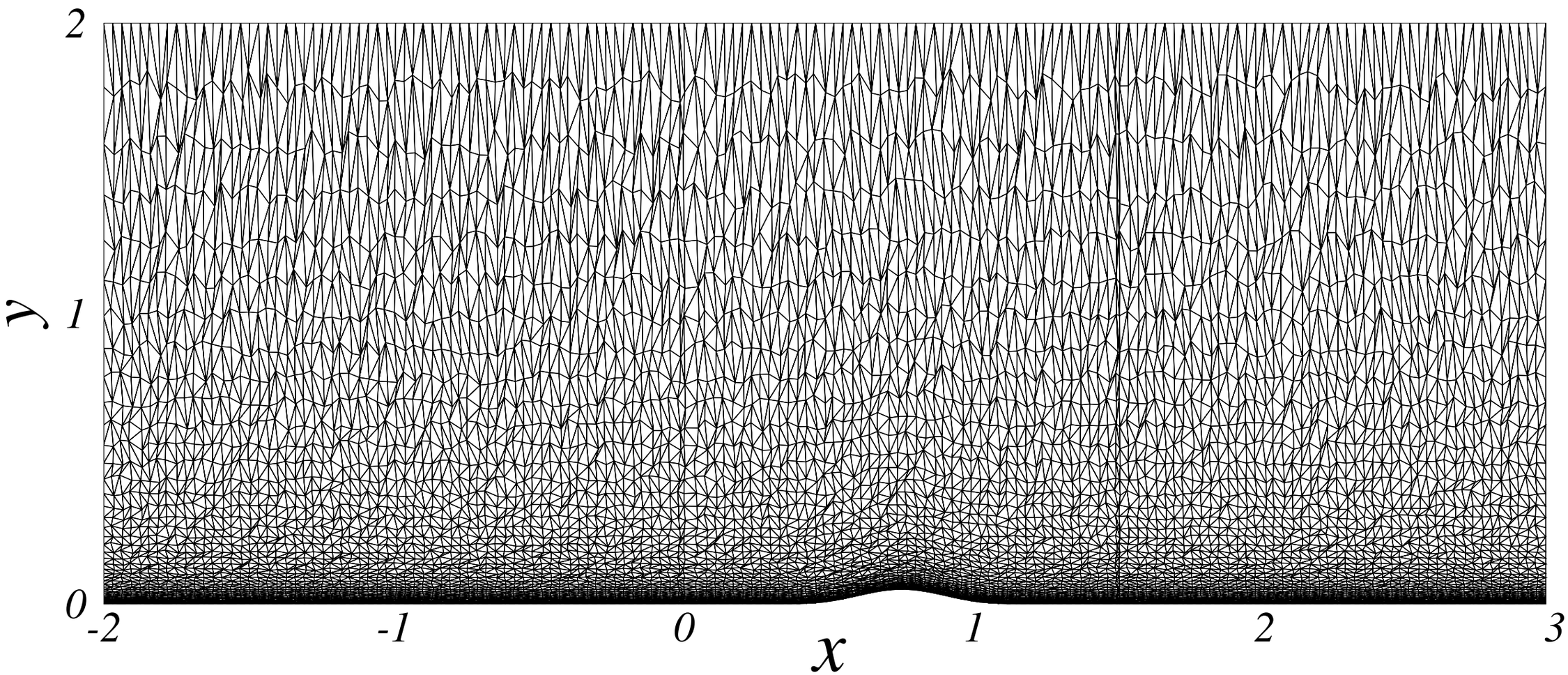
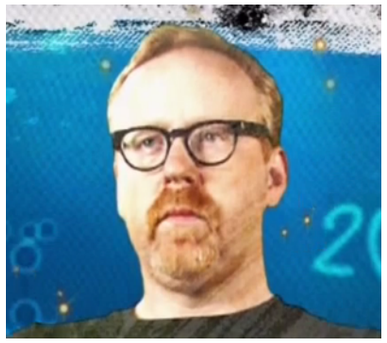
Subsonic Flow over a Bump

Mach 0.3, Re=100, Pr=3/4, Grids: 12800, 51200, 204800 nodes.

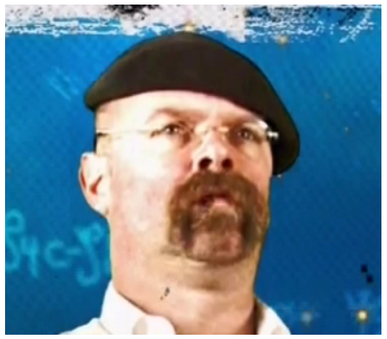


Highly-stretched viscous triangular grid.



Adam:

“ The grid is too skewed.
Can we have a better grid? ”



Jamie:

“ Well, this is a good chance to investigate the CFD myth #1. ”

Nice and smooth viscous grid:

“ Boundary-layer grid must be smooth and orthogonal. “

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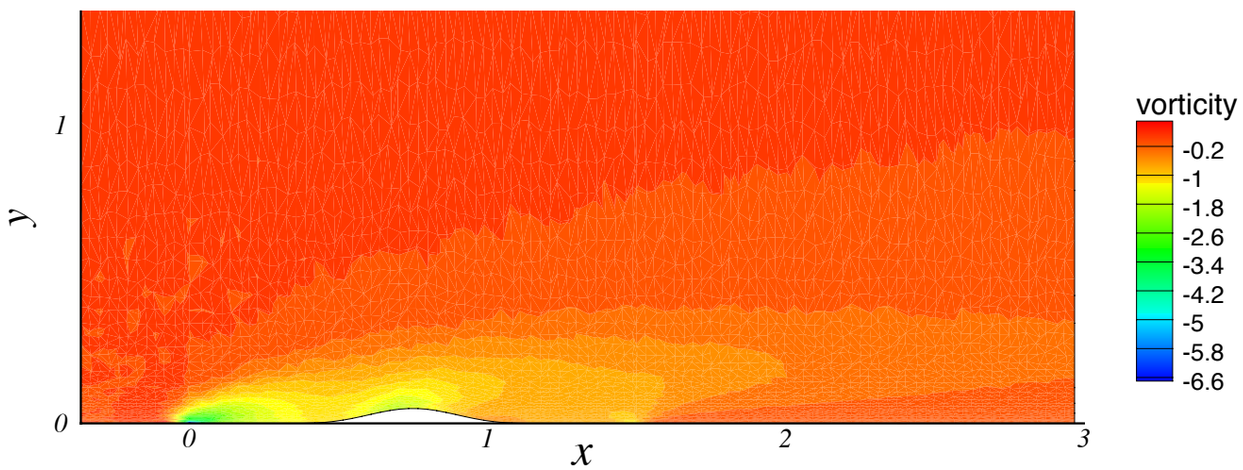
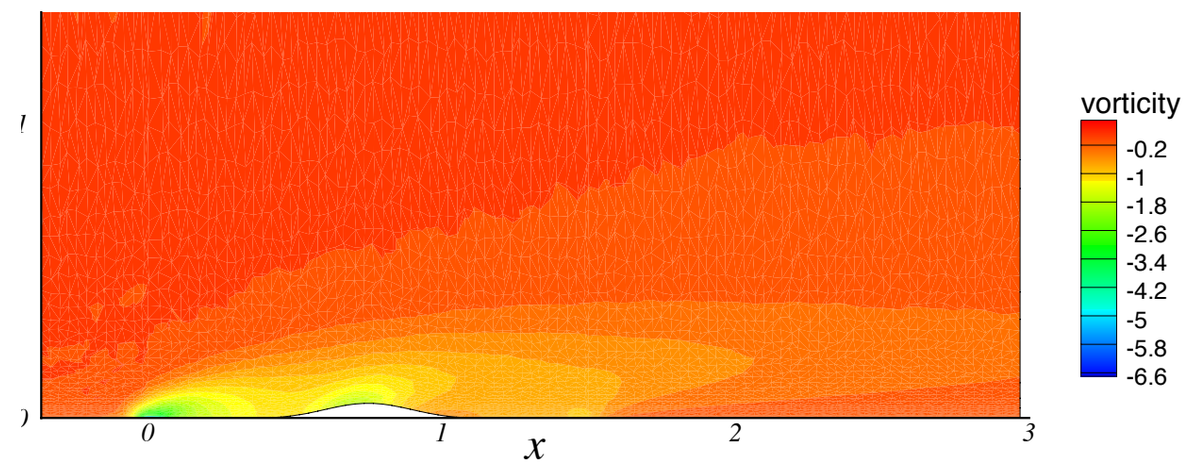
Subsonic Flow over a Bump

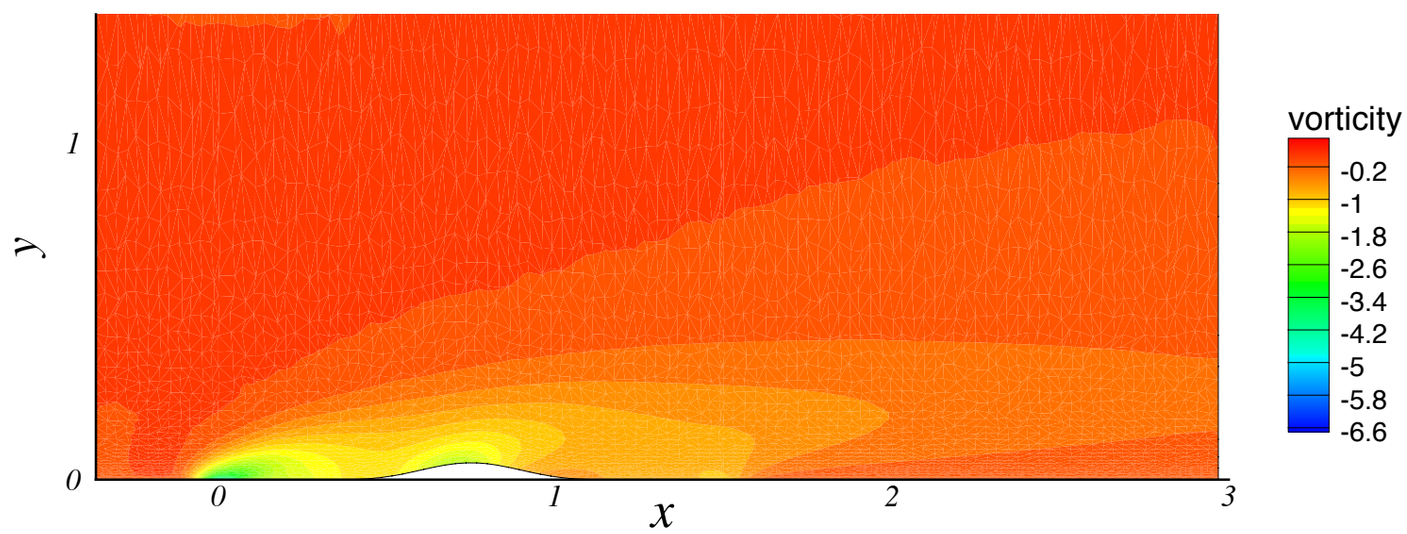
Vorticity Contours



Conventional



Hyperbolic(2nd)



Hyperbolic(3rd)

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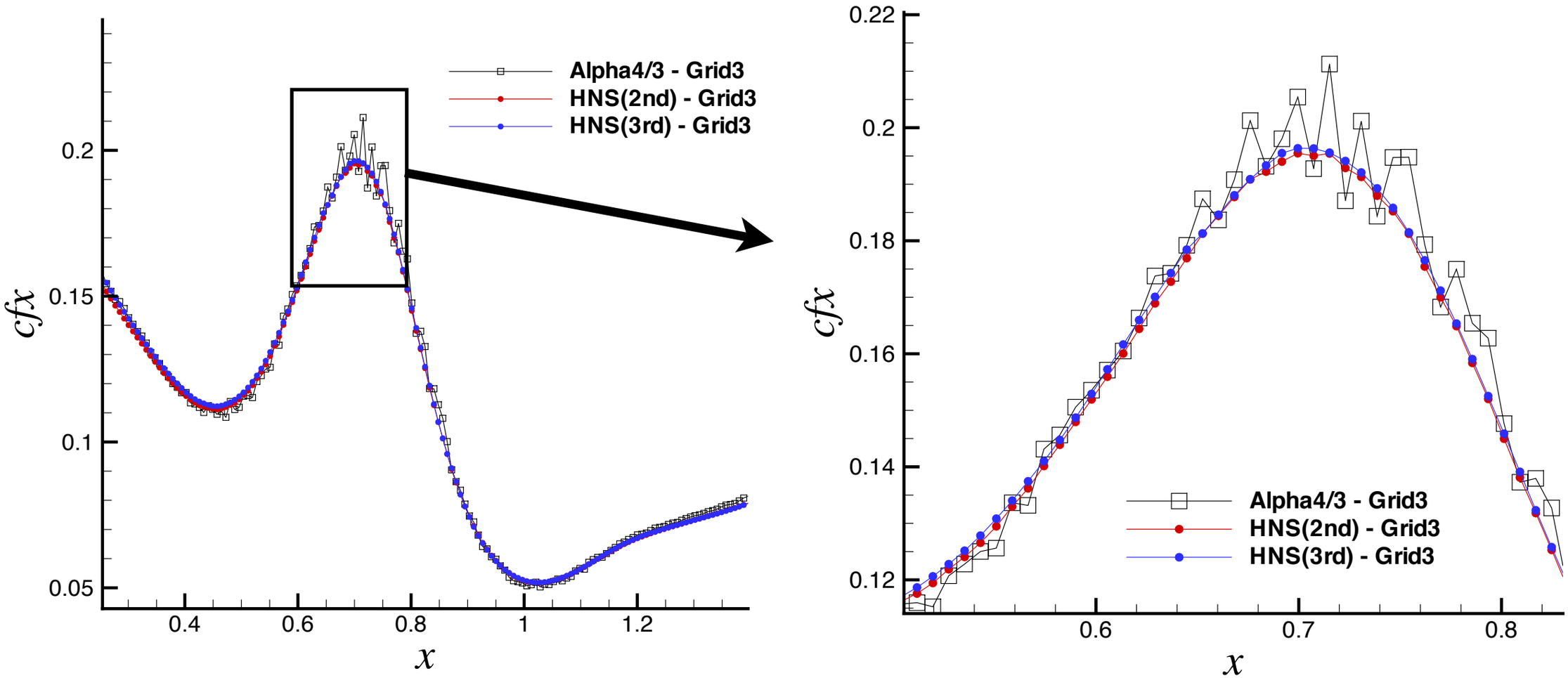
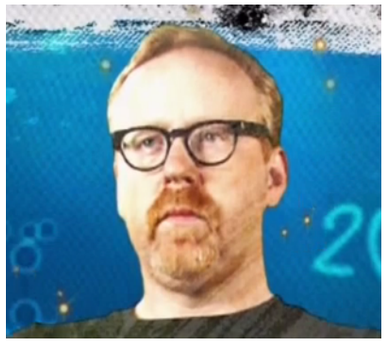
Subsonic Flow over a Bump

Skin Friction Coefficient over the bump



Highly oscillatory distribution by conventional scheme

Very smooth distribution by hyperbolic schemes



Adam:

“ That’s very nice.
But how about the cost? ”

Jamie:

“ Well, you don’t believe this.”

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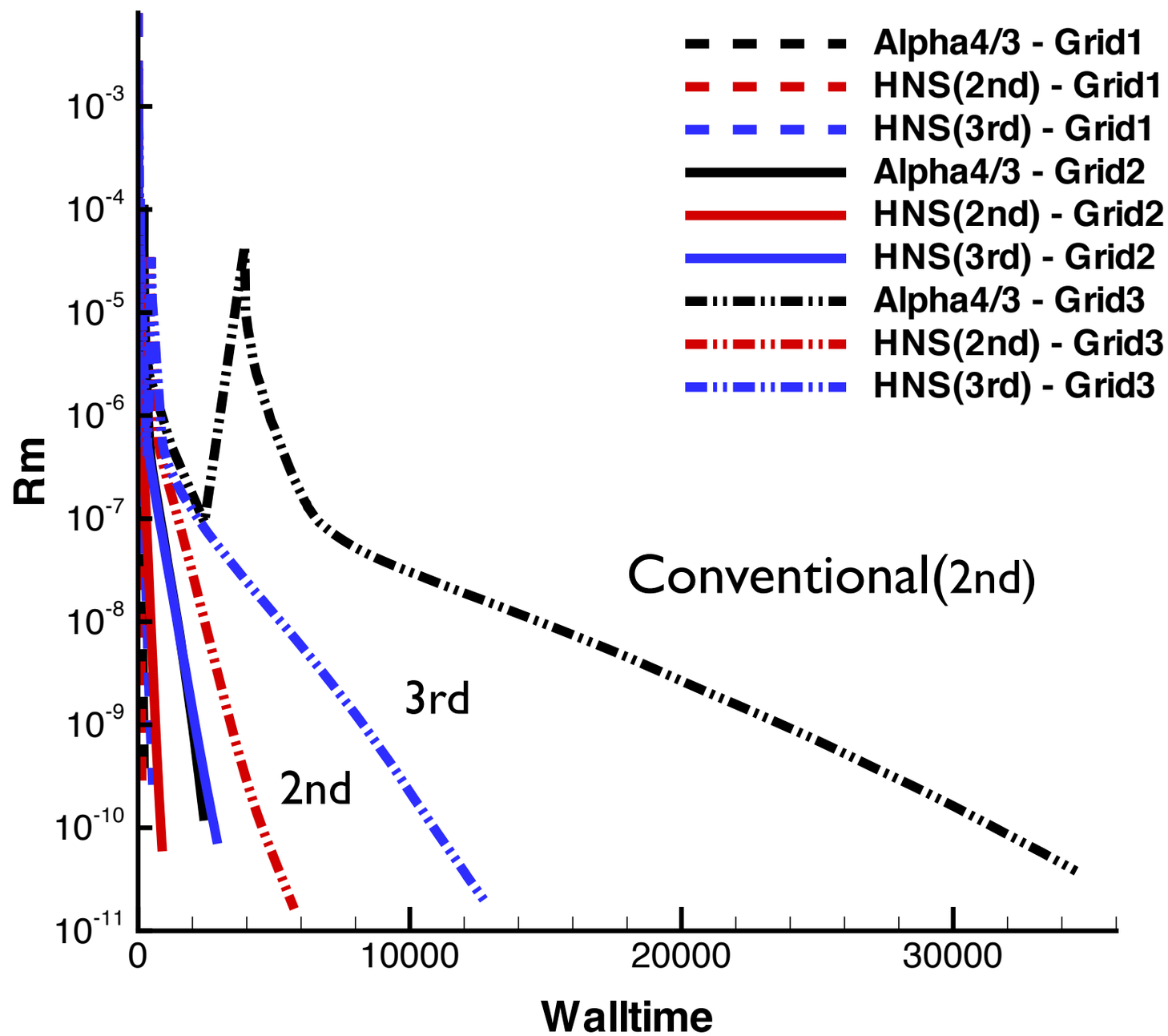
Subsonic Flow over a Bump



Hyperbolic schemes converge faster in CPU than conventional scheme

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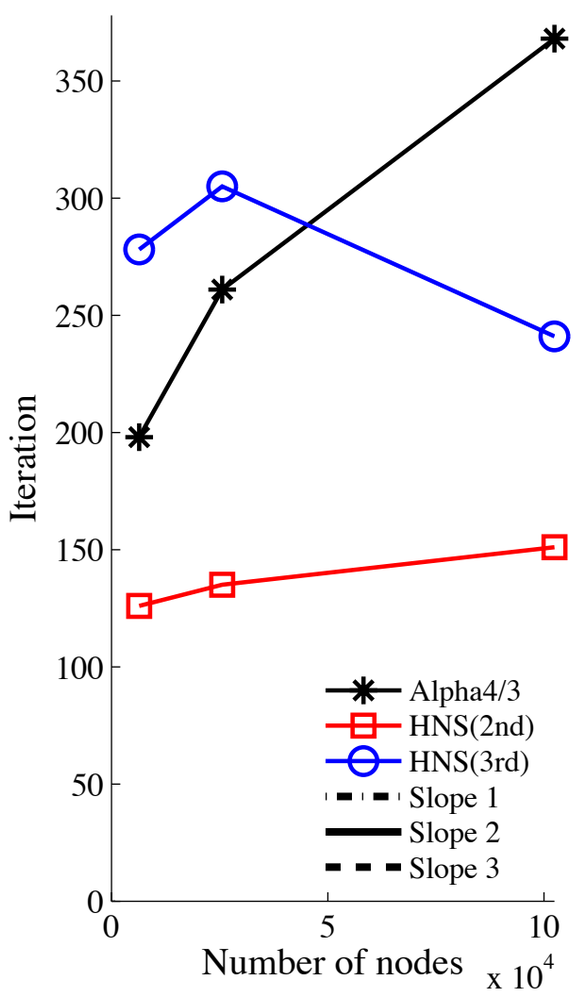
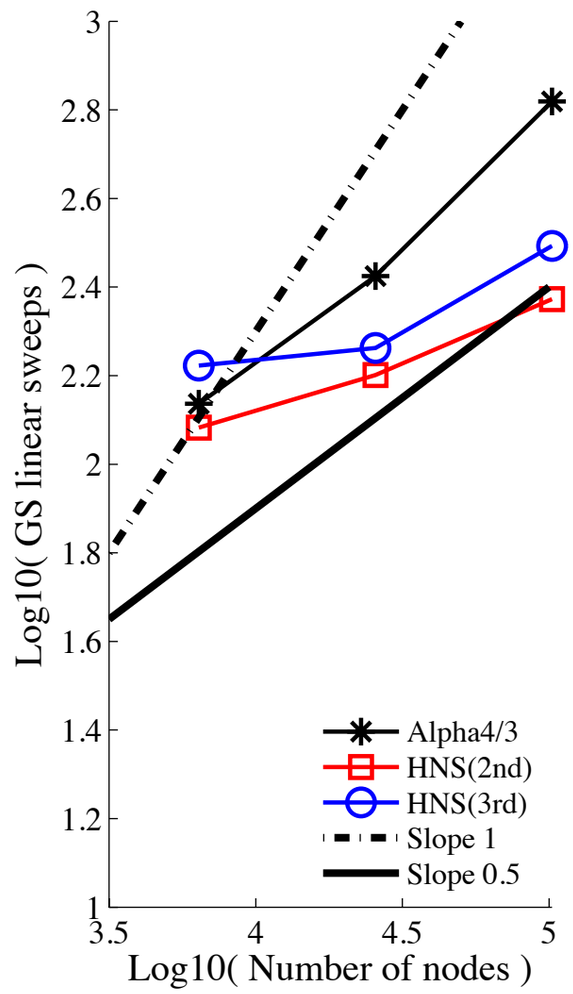
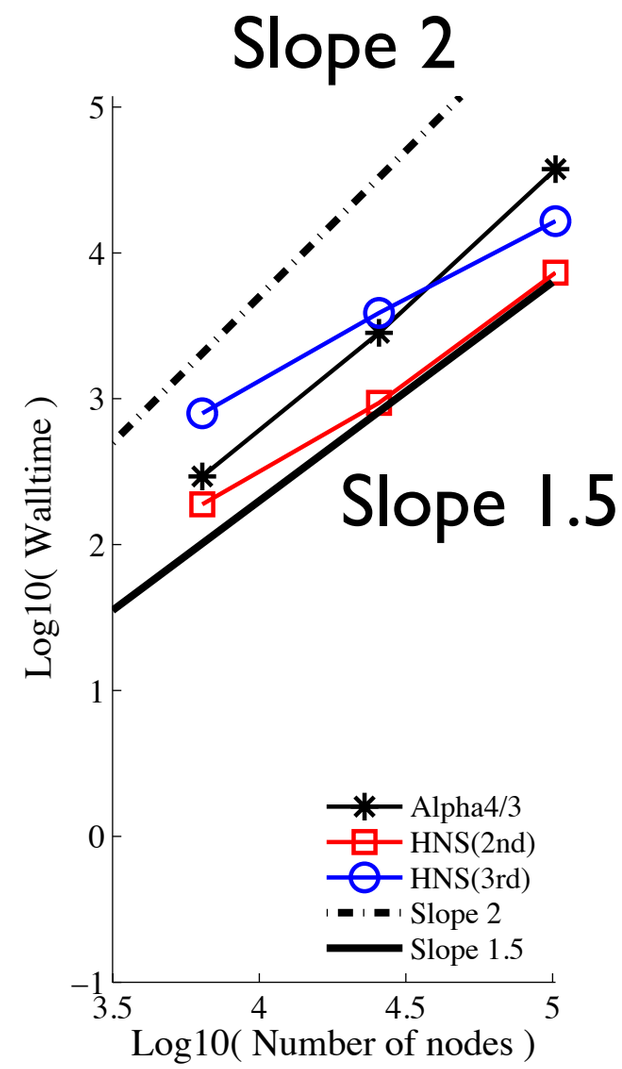
Subsonic Flow over a Bump

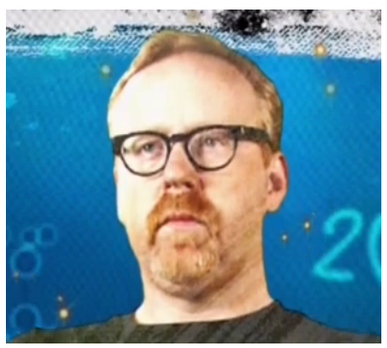
Convergence Histories







Observe the difference in the slopes:
Hyperbolic schemes are intrinsically more efficient.



Adam:

" OK, the speed-up comes largely from the linear relaxations. But typically the number of linear relaxations is fixed for a practical reason."

Jamie:

"My guess is it is still faster because the linear system is better relaxed."

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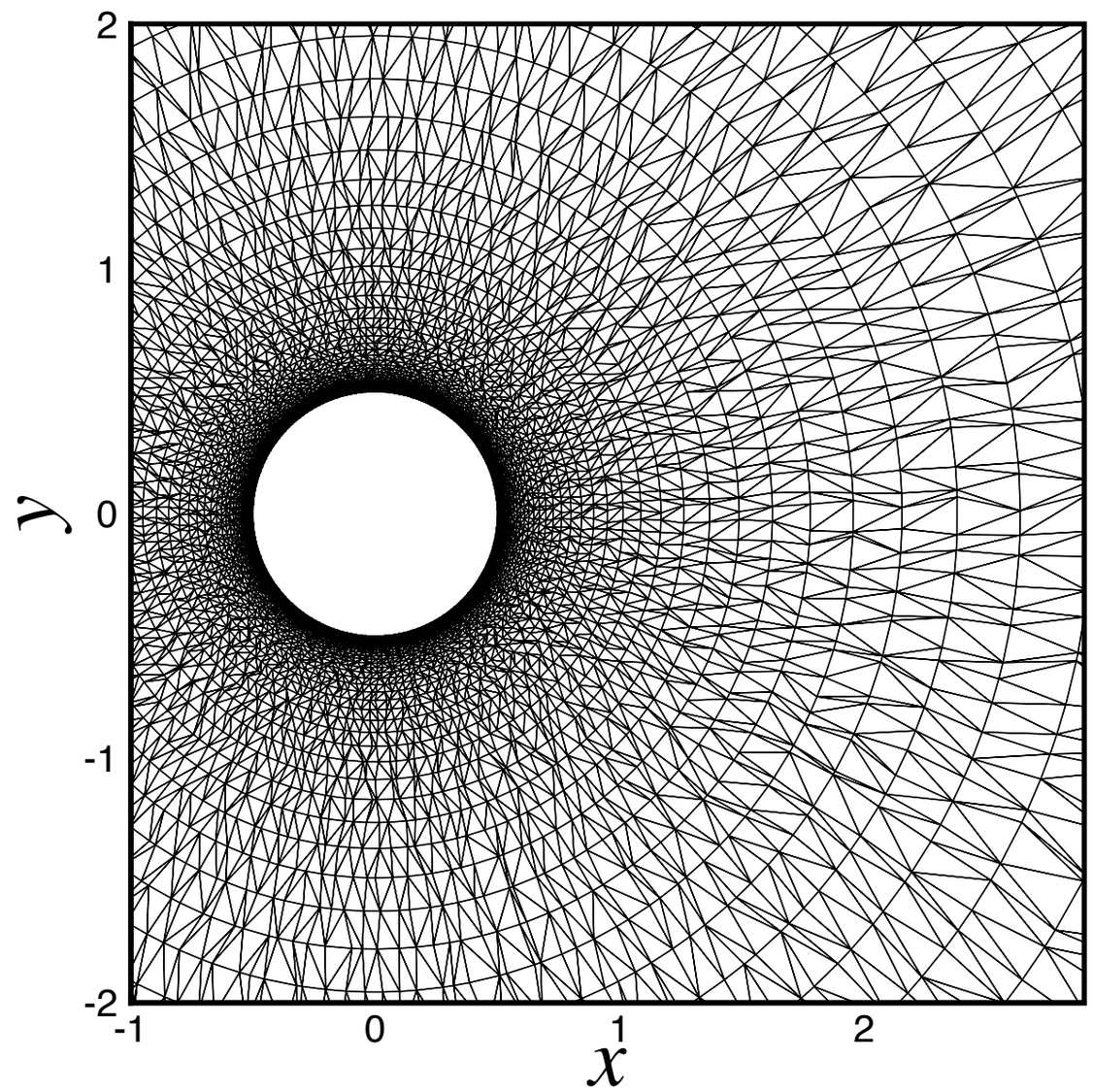
Subsonic Flow over a Cylinder

Mach 0.2, Re=40, Pr=3/4, Grids: 12800, 51200, 204800 nodes.

A fixed number of GS linear relaxations = 100



Outer boundary located at a distance 500xdiameter

Highly-skewed viscous triangular grid, typically not used in practical cases.

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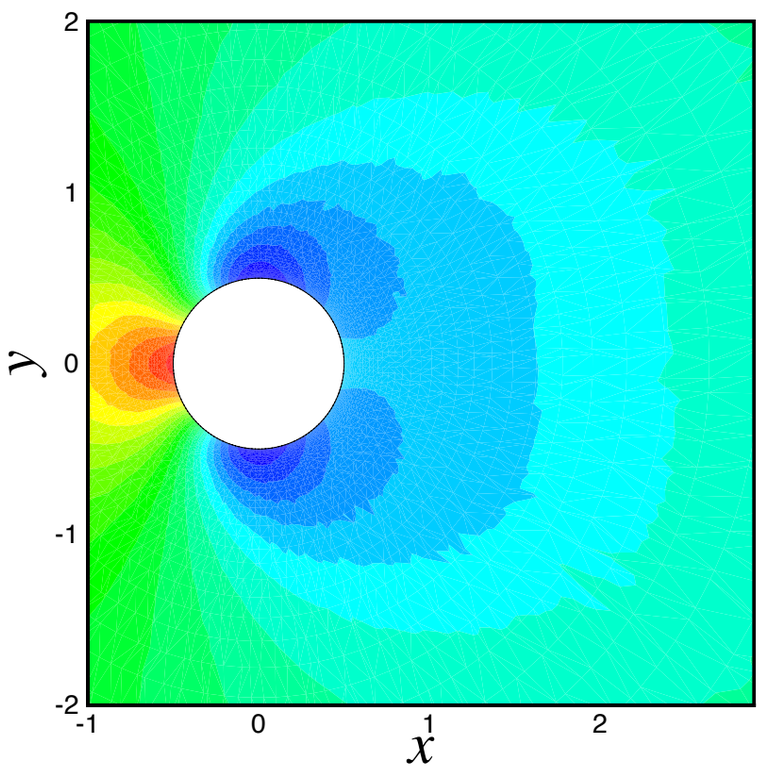
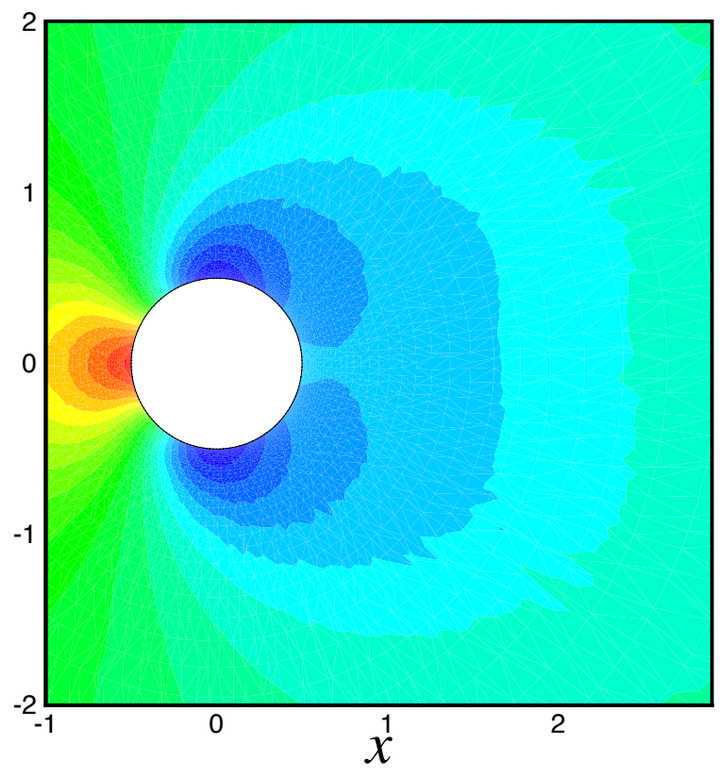
Subsonic Flow over a Cylinder

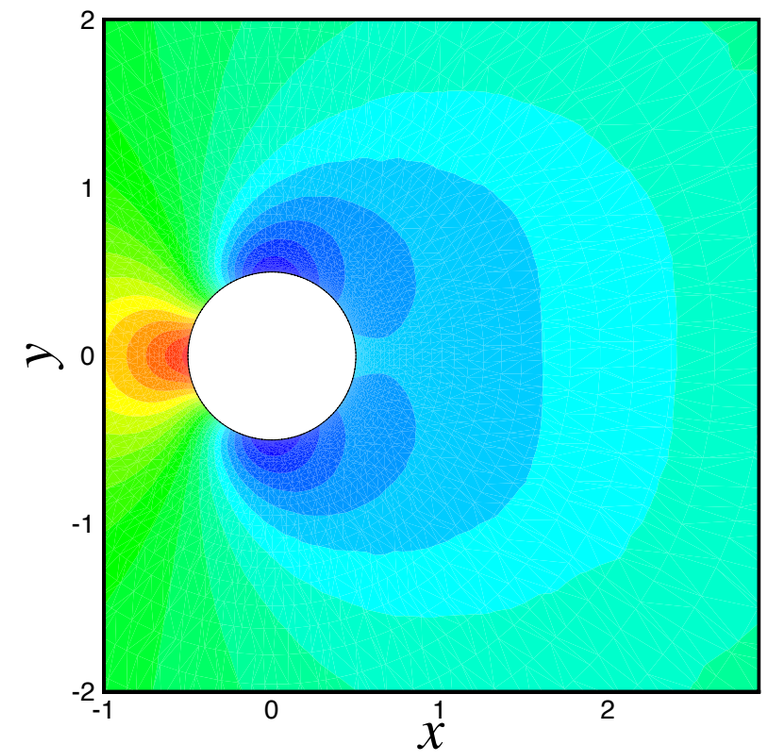
Pressure Contours



Conventional



Hyperbolic(2nd)



Hyperbolic(3rd)

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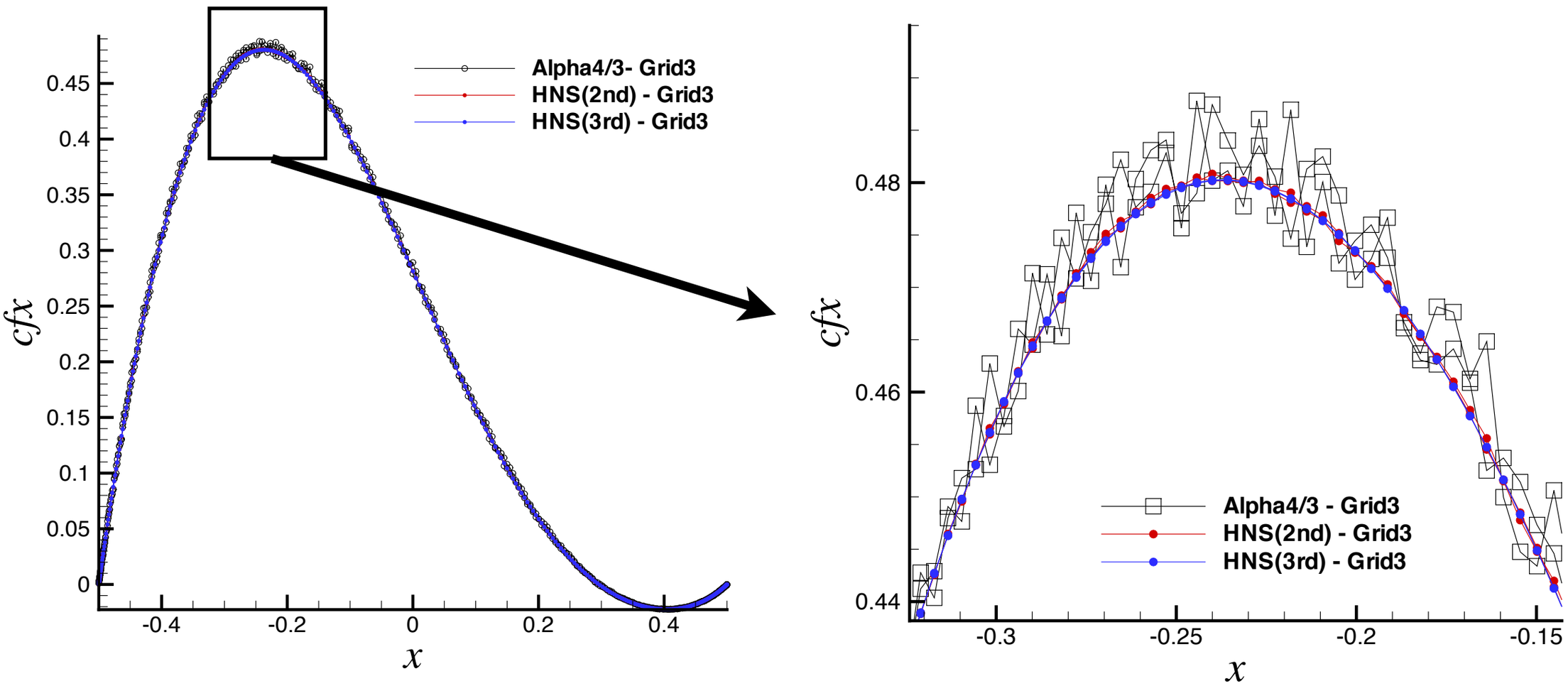
Subsonic Flow over a Cylinder

Skin Friction Coefficient over the cylinder



Highly oscillatory distribution by conventional scheme

Very smooth distribution by hyperbolic schemes

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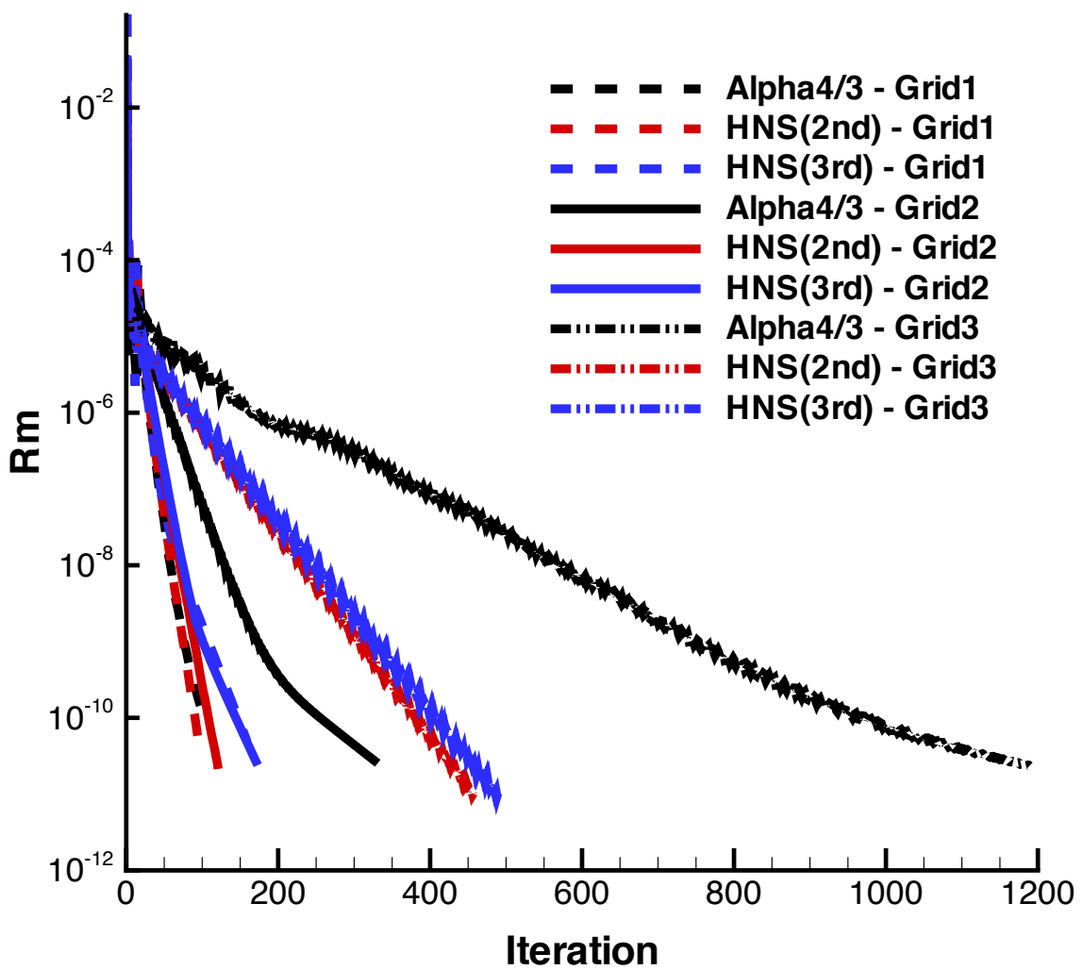
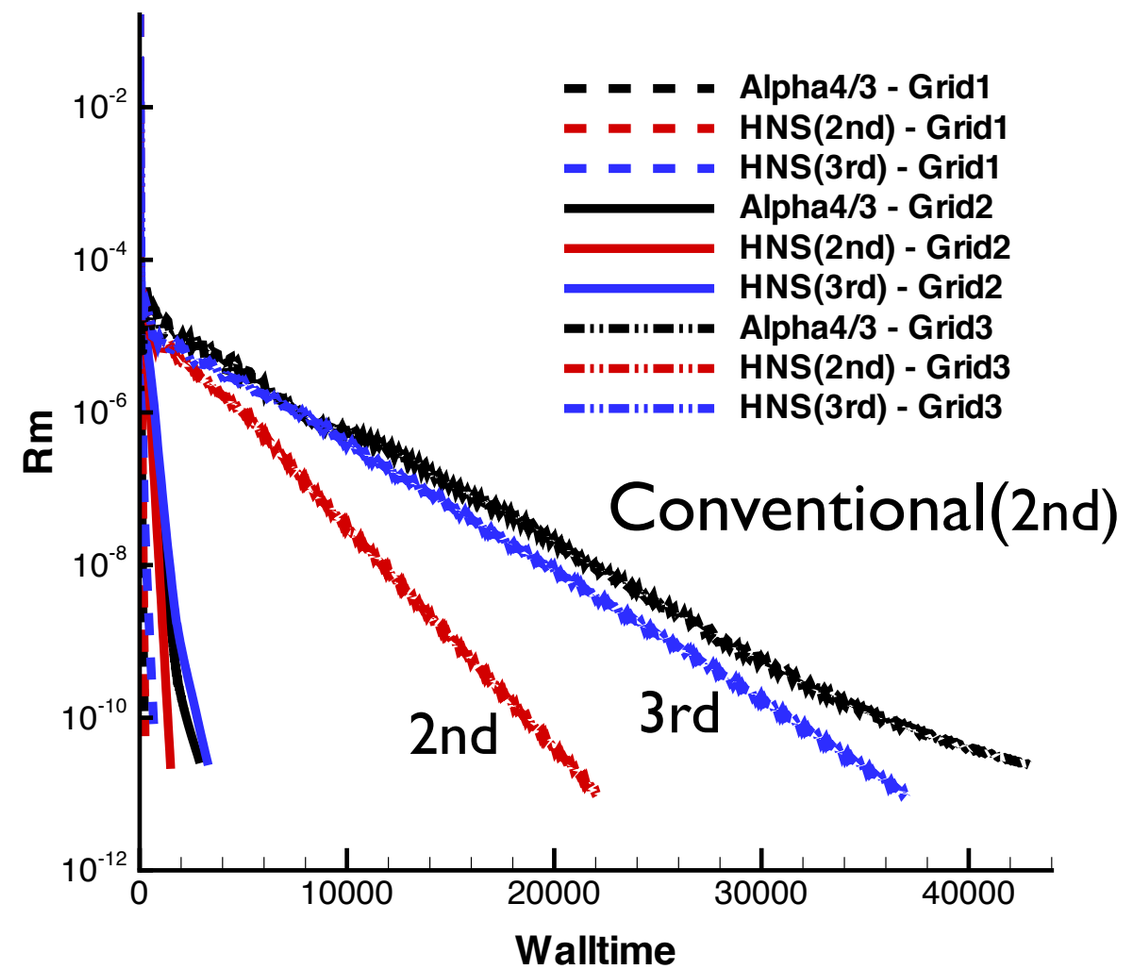
Subsonic Flow over a Cylinder

Convergence Histories





Hyperbolic schemes still converge faster in CPU than conventional scheme even with a fixed number of GS relaxations (100)

Conclusions

1st/2nd/3rd-order implicit upwind FV schemes constructed for the Navier-Stokes equations on irregular grids.

1st/2nd/3rd-order accurate viscous and heat fluxes.

Consistent Jacobian (Newton for 1st-order)

Hyperbolic schemes are intrinsically more accurate and efficient than conventional schemes.

Remarks

Explicit time-stepping is not possible for unsteady:

Unsteady scheme must be implicit or space-time.

(See NASA-TM2014 and CF2014 for **unsteady** schemes.)

More memory required:

Potentially, 3 times more, and for implicit solver, somewhere between 3 and 9 times more memory is required in 3D: 3x5 residuals and (3x5)X(3x5) Jacobian.

Future Work

- More extensive numerical experiments.
- New approach to accurate velocity gradients
- High-Reynolds turbulent flows, and shock waves.

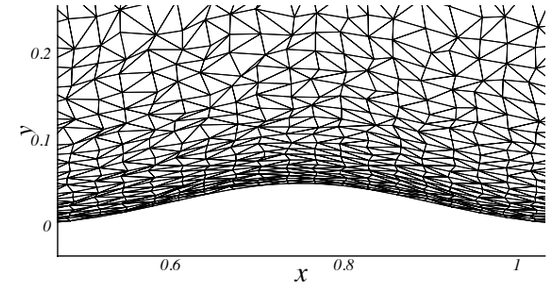
Many others...

Attractive features encourage further study:

- Smooth and accurate viscous stresses and heat fluxes on irregular grids

- Speed up the current state-of-the-art CFD codes with high-order accuracy

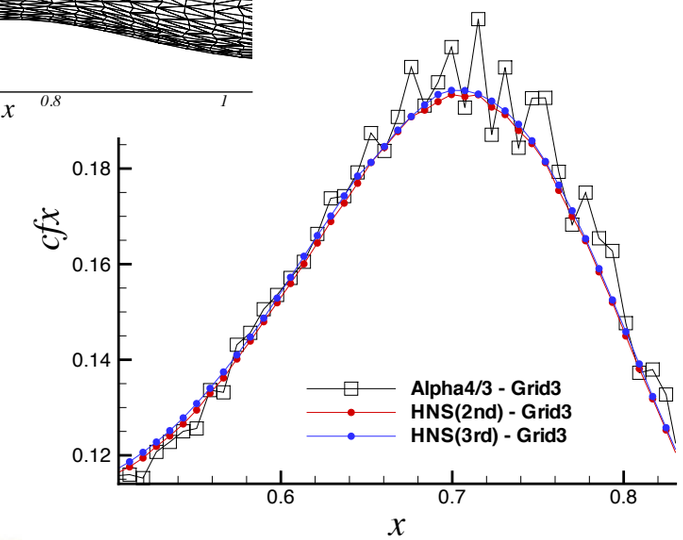
CFD Myth #1



Nice and smooth viscous grid:

“ Boundary-layer grid must be smooth and orthogonal. “

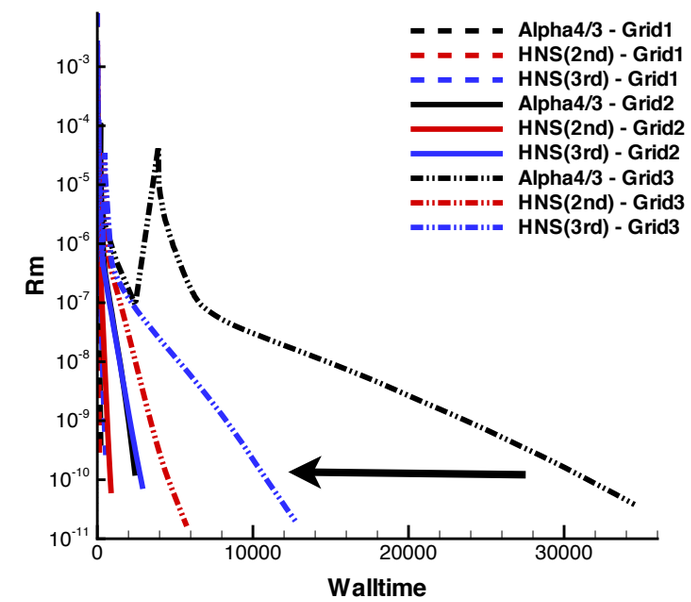
Arbitrary grid-adaptation possible inside BL.

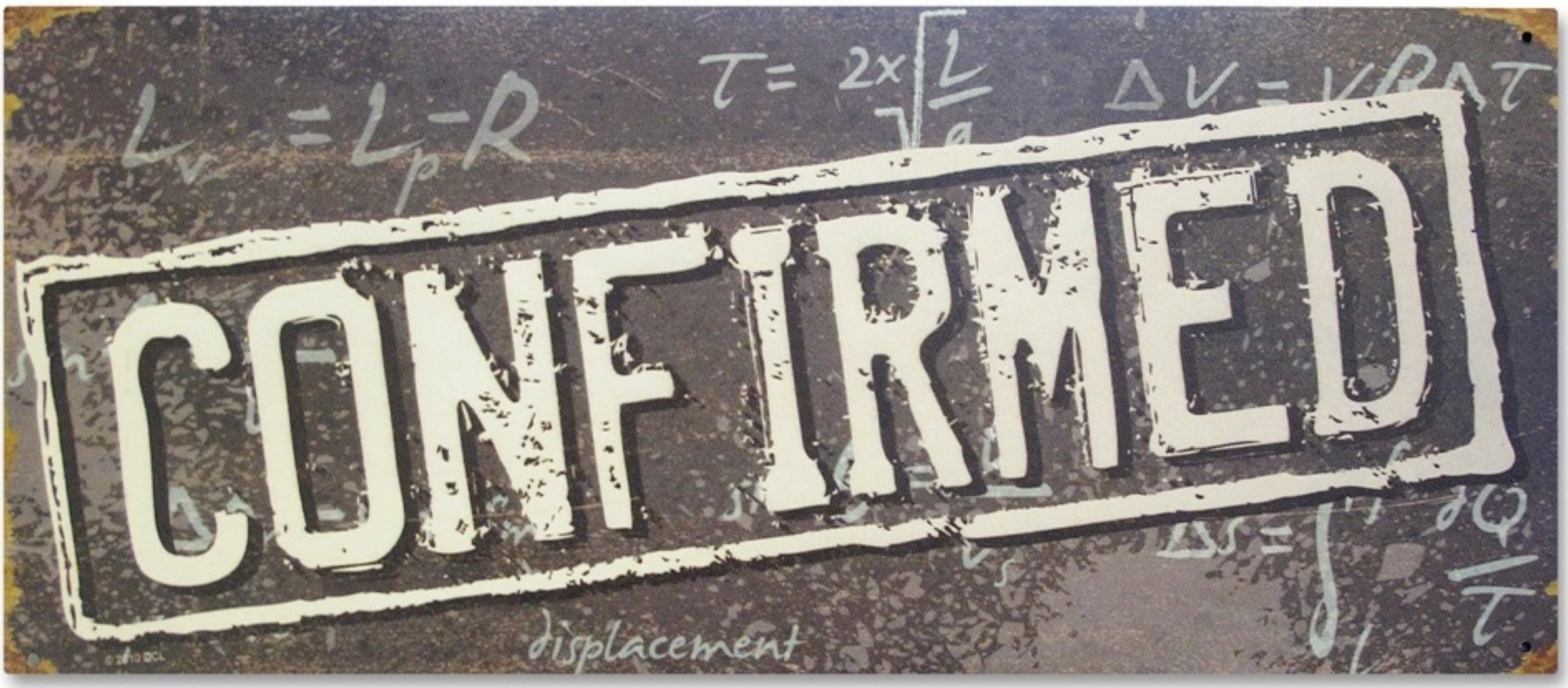


CFD Myth #2

Impossible High-Order Scheme:

“ There exists a 3rd-order scheme that is less expensive on a given grid than a 2nd-order FV scheme widely used today. “





“Extension to 3D is straightforward.”

To be investigated...

“ When the guys don't get what they want, they just keep on going until they do. "