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On High-Order Fluctuation-Splitting Schemes for Navier-Stokes Equations

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1 Introduction

The Navier-Stokes equations express a balance of advection and diffusive terms; the former are always dominant far from the solid surfaces, and the latter dominate close to the surface. One needs numerical methods that are accurate in both of these limiting situations, and everywhere between. In the purely diffusive limit, it is appropriate to use central differencing, and to appeal to minimization principles. For the opposite limit, upwinding or artificial dissipation must be used to provide stability, and minimum principles do not apply. There is a lack of uniformly applicable design principles, and some dissatisfaction with current practice [7].

Here we will examine a simpler problem that raises the same issues, scalar advection-diffusion, surveyed comprehensively by Morton [6]

$$u_t + a\,u_x + b\,u_y = \nu\,(u_{xx} + u_{yy}), \tag{1}$$

We define Reynolds numbers $Re_{(L,h)} = \sqrt{a^2+b^2}(L,h)/\nu$, where the characteristic length is either a typical length scale L in the problem or the mesh spacing h. We focus on schemes based on multidimensional upwinding, or fluctuation-splitting, because of their reduced sensitivity to mesh quality for inviscid flow [1, 2]. This derives in part from the fact that the advection schemes preserve polynomial solutions of certain order on arbitrary grids.

These schemes are based on nodal variables, leading to cell-based residuals that are distributed to the nodes. For purely elliptic problems, there is a superficially similar method that distributes gradients, rather than residuals. It is equivalent to the classical Galerkin discretization, and is second-order accurate. However, simply adding this to a successful advection scheme may yield only a first-order method for (1) at finite Reynolds number.

2 Losing $\mathcal{O}(h^2)$

It has been common to construct fluctuation-splitting Navier-Stokes codes by adding a Galerkin discretization of the viscous term to a second-order Euler code. [3, 10, 11, 12]. This leads to the following discretization of (1),

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$$\frac{du_j}{dt} = \sum_{T \in \{T_j\}} \left[\beta_j^T \phi_{\text{adv}}^T - \frac{\nu}{2} (\text{grad}\, u)^T \cdot \mathbf{n}_T \right], \tag{2}$$

where the right hand side collects updates contributed from the triangles $\{T_j\}$ around the node j. The first term distributes, according to some advection scheme with weights β_j^T, the advective residuals ϕ_{adv}^T, and the second term evaluates the standard Galerkin discretization of the Laplacian.

For the above scheme on a grid obtained by inserting diagonals into a uniform Cartesian grid with the spacing h, we present the Taylor expansion of the RHS for an arbitrary function. The result is, after some manipulation,

$$\begin{aligned} \mathcal{TE} = r &- \frac{\nu h}{2a} \left(a\partial_x + b\partial_y \right) \left(u_{xx} + u_{yy} \right) \\ &- \frac{h^2}{6} \left(a u_{xxx} + b u_{yyy} + 3b u_{xyy} + 3b u_{xxy} \right) + \frac{\nu h^2}{12} \left(u_{xxxx} + u_{yyyy} \right) + \mathcal{O}(h^3) \end{aligned}$$

where $r = -(au_x + bu_y) + \nu \left(u_{xx} + u_{yy} \right)$ which vanishes for exact steady solutions. The last term on the first line arises from interaction between the advective and diffusive effects, and is generally of order h, except in the limiting cases of very large or very small Re_L.

To confirm this analysis, we set up a test problem in a square domain $(0 < x < 1, 0 < y < 1)$ with the exact solution

$$u = -\cos 2\pi\eta \; \exp\left[0.5\xi(1 - \sqrt{1 + 16\pi^2\nu^2})/\nu)\right]$$

with $\xi = ax + by$, $\eta = bx - ay$. Solutions for $\nu = 0.1$ are computed on a series of uniform grids with spacings $h = 0.1 \times 2^{-n}, n = 0 \ldots 5$. We also analysed and tested a version of this scheme, with third-order gradient correction [9, 8]. This also gave a first-order error at low Reynolds number.

Figure 1 (a) and (b) show the convergence of the numerical errors in the L_2 norm. The 1st-order behavior is obvious for fine grids; the sudden transition between orders of accuracy is striking. The third example (c) is 2nd-order throughout; this scheme is the outcome of the next section.

3 Preserving $\mathcal{O}(h^2)$

To achieve uniform 2nd-order accuracy for the advection-diffusion equation, we follow the philosophy **[2]** that leads to high-order advection schemes: evaluate the measure of the error (*fluctuation*) over the element and make this drive changes to each nodal solution (*distribution*). That is, the fluctuation for the whole equation should be distributed:

$$\frac{du_j}{dt} = \sum_{T \in \{T_j\}} \alpha_j^T \phi^T \tag{3}$$

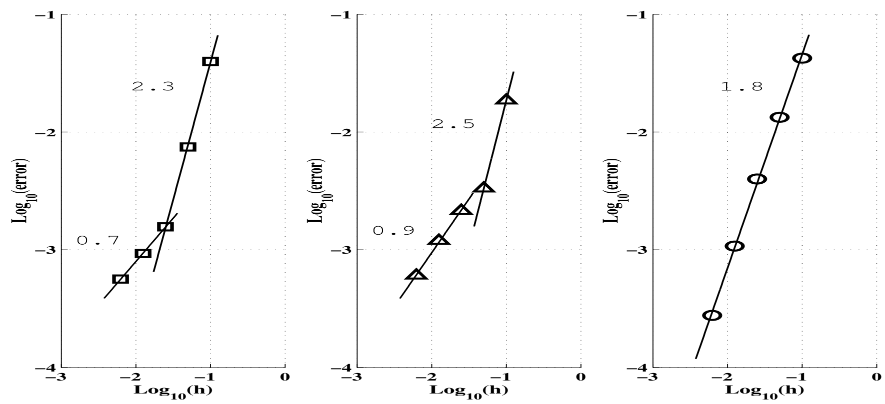

Fig. 1. Error Convergence: (left) LDA advection plus Galerkin diffusion; (center) The same but with gradient corrections applied to both terms; (right) unified LDA distribution of both residuals.

where

$$\phi^T = \int_T [-(a\,u_x + b\,u_y) + \nu\,(u_{xx} + u_{yy})]\;dxdy. \tag{4}$$

Eqn (2) is not of this form. This way, we can make sure that nothing happens if the fluctuation is zero. The viscous contribution to the residual needs to be evaluated cell-by-cell, combined with the advective residual, and then the combination is distributed to the nodes. In the previous section, the viscous residual was evaluated directly at the vertices (the second term of (2)).

In order to carry out this strategy, the viscous stresses (gradients) need to be available at the vertices. Either they can be computed there from the adjacent cells (for example by the Green-Gauss formula, or by least-squares fitting) or stored there as part of a Hermitian representation of the solution. Caraeni and Fuchs[4] developed a scheme employing reconstruction for which they claim third-order accuracy, but their distribution is purely advective, and is not appropriate in the diffusive limit. To remedy this we may use

$$\alpha_j^T = \left(\beta_j^T + \frac{k}{3\,Re_h}\right)/(1 + \frac{k}{Re_h}) \tag{5}$$

where k is some constant. This becomes, for diffusion-dominated flows, a simple isotropic diffusion. The truncation error of this scheme with the Green-Gauss reconstruction is

$$\mathcal{TE} = r + \frac{h}{2a(1+\frac{k}{Re_h})}(a\partial_x + b\partial_y)\,r + \mathcal{O}(h^2) \tag{6}$$

There is still a contribution proportional to h, but this now multiplies the total residual, which vanishes for an exact steady-state solution, making this method uniformly 2nd-order accurate even with $k = 0$. It is this scheme that produced the result in Figure 1 (c). However, we found that obtaining uniformly higher-order accuracy, as in the next section, required us to take $k \neq 0$.

In fact, for pure diffusion, the method is then fourth-order on the regular triangularization employed. Knowing this, we attempted to construct a 3rd-order scheme (improving on Caraeni and Fuchs[4]) by upgrading the advective part to 3rd-order with the high-order correction[9]. But this again resulted in the same form of the truncation error as (6), i.e. only 2nd-order. It became apparent that we were still missing a guiding principle.

4 Beyond $\mathcal{O}(h^2)$

To see what goes wrong, let $r = r_a + r_d$ where $r_a = -(a\,u_x + b\,u_y)$, $r_d = \nu\,(u_{xx} + u_{yy})$. A useful property of many higher-order residual distribution schemes is that their truncation error contains terms proportional to the residual. Thus, if we evaluate the residuals to third order (using corrections obtained from estimated vertex gradients), and then distribute these with central weights ($\alpha_j^T \equiv 1/3$) we obtain the following truncation errors, in the limits of pure advection and pure diffusion.

$$\mathcal{TE}_{\text{adv}} = r_a + \frac{5h^2}{36}(\partial_{xx} + \partial_{xy} + \partial_{yy})\,r_a + \mathcal{O}(h^4) \tag{7}$$

$$\mathcal{TE}_{\text{diff}} = r_d + \frac{h^2}{3}(\partial_{xx} + \partial_{xy} + \partial_{yy})\,r_d + \mathcal{O}(h^4) \tag{8}$$

both of which are 4th-order accurate at their own steady-states ($r_a = 0$ or $r_d = 0$). However the truncation error for their sum is what is relevant in solving the advection-diffusion equation, and that is

$$\mathcal{TE} = r_a + r_d + \frac{h^2}{3}(\partial_{xx} + \partial_{xy} + \partial_{yy})\,(\frac{5}{12}\,r_a + r_d) + \mathcal{O}(h^4) \tag{9}$$

which does not vanish for $r = 0$. This leads only to a 2nd-order method, due to incompatible discretizations of advective and diffusive fluctuations. It is noteworthy that accuracy in the two limits does not guarantee accuracy in intermediate cases. A discretization method that does have this property has been developed by Lerat and Corre [5] for structured grids, but at the present time it is not clear to us how to construct an unstructured version.

The approach we have taken here may not be necessary, and may be overelaborate. We avoided the problem of finding compatible discretizations for first- and second-order derivatives by writing the governing equation as a first-order system. We introduce $p - u_x$ and $q = u_y$ as additional unknowns and consider the first-order version of (1)

$$u_t + a\,u_x + b\,u_y = \nu\,(p_x + q_y) \tag{10}$$

$$p - u_x = 0 \tag{11}$$

$$q - u_y = 0. \tag{12}$$

Assume first a piecewise linear variation of u, p, q. By integrating equation (10) over a triangle, we obtain a 2nd-order residual, distributed with coefficients (5) to update u. The residuals of (11), (12), are distributed isotropically ($\alpha_j^T \equiv 1/3$) to update p, q; this minimizes those residuals in an L_2 norm.

To go to higher order, we simply follow the previous works: the fluctuations are made more accurate by adding high-order correction [9, 8], (or equivalently reconstruct gradients $\{u_x, u_y, p_x, p_y, q_x, q_y\}$ at nodes, construct the Hermite interpolation over each edge, integrate the equations as a contour integral to obtain fluctuations). The modified fluctuations are then distributed with the same coefficients as the 2nd-order version.

The truncation error of the fourth-order version is, for (10)

$$\mathcal{TE} = r + c_1 h + c_2 h^2 + c_3 h^3 + \mathcal{O}(h^4) \tag{13}$$

where

$$c_1 = \frac{a r_x + b r_y}{2a}$$
$$c_2 = -\frac{1}{36a}\left\{(6a + 5\nu a) r_{xx} - (3a + 3b + 5\nu a) r_{yy} - (5\nu a + 6b + 3a) r_{xy}\right\}$$
$$c_3 = \frac{1}{24a}\left\{a r_{xxx} + (a + b)(r_{xyy} + r_{xxy}) + b r_{yyy}\right\},$$

The truncation error for (11) is

$$\mathcal{TE}_p = r^p + \frac{5}{36}(r^p_{xx} + r^p_{xy} + r^p_{yy}) h^2 + \mathcal{O}(h^4), \tag{14}$$

and similarly for (12). Therefore the method is 4th-order accurate for u, p, and q at steady-state ($r = 0,\ r^p = 0,\ r^q = 0$).

The schemes just described were applied for the test problem in Section 2, and the results are shown in Figure 3. Clearly we have achieved 2nd-order with the base scheme, 3rd-order with the Green-Gauss gradient reconstruction, 4th-order with a quadratic gradient reconstruction. We remark that the 3rd-order accuracy is only due to poor performance of the Green-Gauss reconstruction near boundaries.

Finally, we obtained results from the fourth-order method on a sequence of fully unstructured grids. As hoped, the convergence was rapid, with an experimental order of accuracy of about 3.5 (see Figure 5).

5 Concluding Remarks

This paper has shown that uniformly second-order fluctuation-splitting schemes can be developed for the advection-diffusion equation by distributing the full residual with proper distribution coefficients. We were able to go beyond second-order by computing the slopes as independent unknowns. All of the ideas should be readily extendable to the Navier-Stokes system.

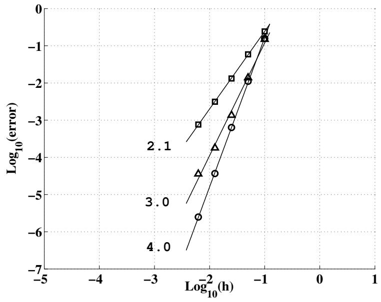

Fig. 2. Error Convergence: the numbers indicate slopes determined by the least-squares fit

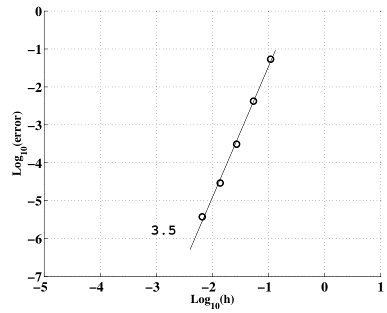

Fig. 3. Results for unstructured grids, each with the same number of nodes as in the regular counterpart

In future work we need to study how these ideas can be combined with the nonlinear distribution schemes needed for monotone shock-capturing, It is likely that a successful distribution coefficient involves the (perhaps local) Reynolds number as a parameter. Second, instead of the high-order correction method to solve the first-order system, we may introduce additional degrees of freedom at the midpoint of each edge to achieve 4th-order accuracy without a need for gradient reconstruction. This type of high-order extension was studied in [2] for the advection equation.

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