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# Third-Order Inviscid and Second-Order Hyperbolic Navier-Stokes Solvers for Three-Dimensional Unsteady Inviscid and Viscous Flows

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# Outline

- Introduction of edge-based discretization and hyperbolic Navier-stokes methods
- Temporal and spatial discretization method
- Source term discretization for third-order accuracy
- Numerical results
  - Inviscid moving vortex
  - Unsteady viscous flow pass a cylinder
- Conclusion remarks and future work

# Edge-Based Discretization

- Widely used in CFD solvers like NASA's FUN3D, Software Cradle's SC/Tetra, DLR Tau code, etc.
- Achieves third-order accuracy for hyperbolic systems on arbitrary simplex-element grids with quadratic least-square (LSQ) methods and linearly-extrapolated fluxes. [1,2,3]


1. Katz and Sankaran, *J. Comput. Phys.*, Vol 230, 2011, pp. 7670-7686
2. Katz and Sankaran, *J. Sci. Comput.* Vol. 51, 2012, pp. 375-393
3. Diskin and Thomas, AIAA Paper 2012-0609

# Hyperbolic Navier-Stokes Method

- Conventional NS has 5 variables $(\rho, \mathbf{v}, p)$
  Hyperbolic NS has 20 variables $(\rho, \mathbf{v}, p, \mathbf{r}, \mathbf{g}, \mathbf{q})$
  15 gradient variables. $(\mathbf{r}, \mathbf{g}, \mathbf{q}) \longrightarrow \nabla(\rho, \mathbf{v}, p)$
- Same order of accuracy can be achieved for both solutions and gradient quantities (e.g. viscous stress, vorticity, heat fluxes) on irregular grids.
- Accurate and non-oscillatory gradients can be obtained on highly distorted grids.
- Hessian can be obtained with first-order accuracy on irregular grids.

# Hyperbolic Navier-Stokes Method
(cont)

- Use same edge-based discretization for inviscid and viscous terms; can achieve third-order accuracy for the inviscid terms by second-order algorithm.
- Successfully implemented in existing 3D CFD solvers and demonstrated the above advantages of HNS method[4,5]

4. Nakashima, Watanabe, Nishikawa, AIAA Paper 2016-1101
5. Liu, Nishikawa, AIAA Paper 2016-3969

**Hyperbolic method website:**
**http://hiroakinishikawa.com/fohsm**

# Third-order-inviscid Edge-based Schemes in FUN3D

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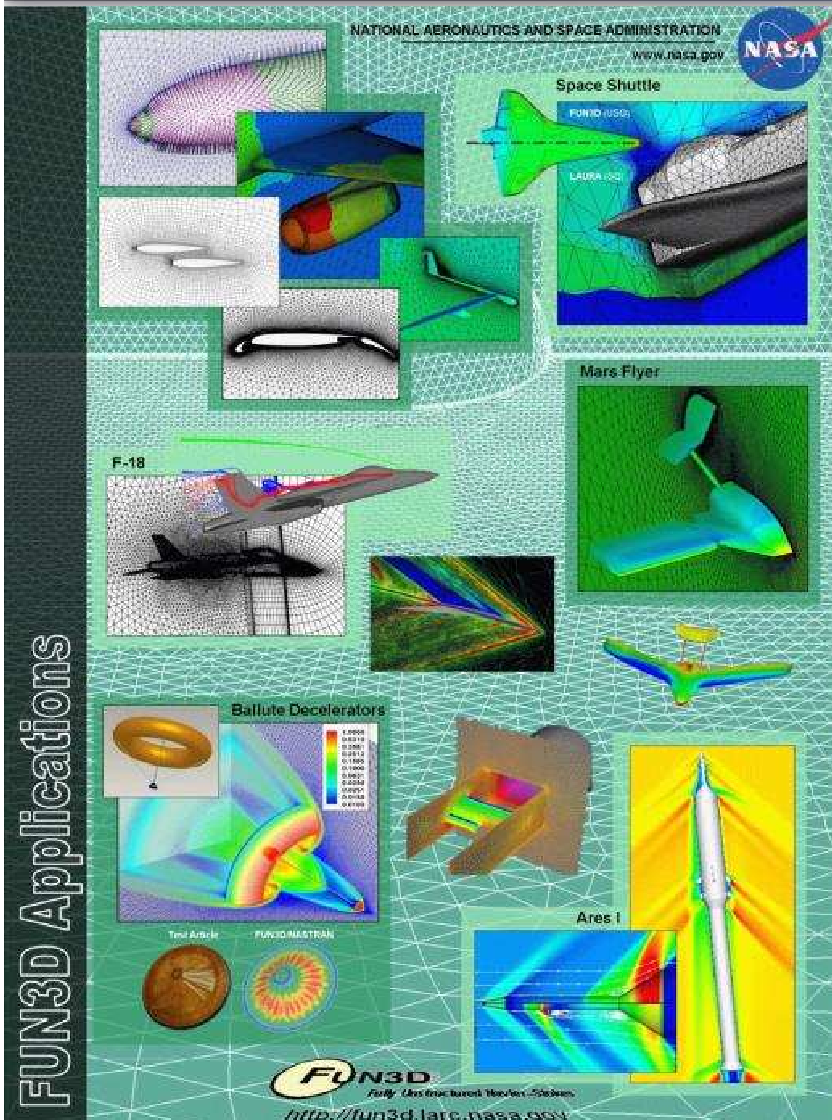



- FUN3D, developed by NASA LaRC, is chosen as the CFD solver for the implementation of the third-order-inviscid scheme.
- FUN3D-i3rd, the third-order-inviscid scheme plus the Galerkin discretization for viscous terms.
- FUN3D-HNS20 (HNS20), the third-order-inviscid scheme with Hyperbolic Navier-stokes viscous discretization.

# Discretization Methods

Governing Equations:

$$\mathbf{P}^{-1}\partial_\tau \mathbf{U} + \mathrm{div}\mathbf{F} = \mathbf{S} - \partial_t \mathbf{U}$$

Temporal Discretization, backward-difference formulas (BDF)[6]

$$\partial_t \mathbf{U} = \alpha \mathbf{U} + \sum_{k=1}^{m} \alpha_{n-(k-1)} \mathbf{U}^{n-(k-1)}$$

$$\boxed{\mathbf{P}^{-1}\partial_\tau \mathbf{U} + \mathrm{div}\mathbf{F} = \mathbf{S} - \mathbf{G}}$$

$$\mathbf{G} = \alpha \mathbf{U} + \sum_{k=1}^{m} \alpha_{n-(k-1)} \mathbf{U}^{n-(k-1)}$$

6. Vatsa, Carpenter, AIAA Paper 2005-5245

# Summery of Discretization Methods

## Spatial Discretization:

$$V_j \frac{d\mathbf{U}_j}{d\tau} = -\mathbf{P}_j \left( \sum_{k \in \{k_j\}} \mathbf{\Phi}_{jk} A_{jk} - \mathcal{L}_j(\mathbf{S} - \mathbf{G}) \right)$$

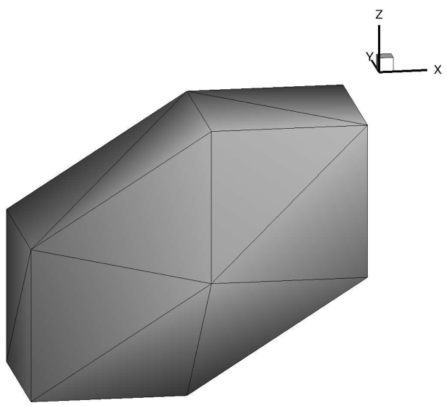
| Scheme | Discretization | | | | | | Jacobian | |
|---|---|---|---|---|---|---|---|---|
| | Inviscid | | | Viscous | | | Inviscid | Viscous |
| | Flux | | LSQ $(\rho, \mathbf{v}, p)$ | Flux | | LSQ $(\mathbf{r}, \mathbf{g}, \mathbf{q})$ | | |
| FUN3D | Roe(2nd) | : 2 | Linear | Galerkin(2nd) | : 1 | None | Van Leer | Exact |
| FUN3D-i3rd | Roe(3rd) | : 3 | Quadratic | Galerkin(2nd) | : 1 | None | Van Leer | Exact |
| HNS20 | Roe(3rd) | : 2 | C-quadratic | Upwind(2nd) | : 2 | Linear | Van Leer | Upwind |

# Source Term Discretization

Compatibility Conditions:

$$\mathcal{T}_j = \mathrm{div}\mathbf{F}^i - \frac{h^2}{12}\left[\partial_{xx}\left(\mathrm{div}\mathbf{F}^i\right) + \partial_{yy}\left(\mathrm{div}\mathbf{F}^i\right) + \partial_{zz}\left(\mathrm{div}\mathbf{F}^i\right) - \partial_{xy}\left(\mathrm{div}\mathbf{F}^i\right) - \partial_{yz}\left(\mathrm{div}\mathbf{F}^i\right) + \partial_{zx}\left(\mathrm{div}\mathbf{F}^i\right)\right] + O(h^3)$$

- Second-order error vanishes because the exact solutions satisfies $\mathrm{div}F^i$=0




$$\mathbf{r}_j = \mathrm{div}\mathbf{F}^i - \mathbf{S} + \mathbf{G}$$

$$\mathcal{T}_j = \mathrm{div}\mathbf{F}^i - \mathbf{S}_j + \mathbf{G}_j - \frac{h^2}{12}\left[\partial_{xx}\mathbf{r}_j + \partial_{yy}\mathbf{r}_j + \partial_{zz}\mathbf{r}_j - \partial_{xy}\mathbf{r}_j - \partial_{yz}\mathbf{r}_j + \partial_{zx}\mathbf{r}_j\right] + O(h^3)$$

- Second-order error vanishes for the exact solutions satisfies $\mathrm{div}F^i$-S+G=0

# Source Term Discretization

- **The compatibility condition on regular grid**
- **The elimination of the first-order truncation error**

In two dimensional (2D):

- Extended Galerkin formula[7]
- Divergence formulation method for source terms[8]

- Requires computations and storage of the second derivatives of the source terms.
- Divergence formulation needs to a careful discretization at boundary nodes.


7. Pinkcock, Katz, *J. Sci. Comput*,  Vol.61, 2014, pp. 454-476
8. Nishikawa, *J. Comput. Phys*. Vol.231, 2012, pp. 6393-6400

# Accuracy-Preserving Source Term Quadrature Formulas for Three-Dimension

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$$\int_{V_j} \mathbf{G}\, dV = \sum_{k \in \{k_j\}} \frac{1}{2}(\mathbf{G}_L + \mathbf{G}_R) V_{jk}$$

$$\begin{aligned}
\partial_{jk} &= \Delta \mathbf{x}_{jk} \cdot (\partial_x, \partial_y, \partial_z) = (x_k - x_j, y_k - y_j, z_k - z_j) \cdot (\partial_x, \partial_y, \partial_z), \\
V_{jk} &= \frac{1}{6}(\Delta \mathbf{x}_{jk} \cdot \mathbf{n}_{jk}), \quad V_j = \sum_{k \in \{k_j\}} V_{jk}, \\
\mathbf{G}_L &= a_L \mathbf{G}_j + b_L \partial_{jk} \mathbf{G}_j + c_L \partial_{jk}^2 \mathbf{G}_j, \\
\mathbf{G}_R &= a_R \mathbf{G}_k + b_R \partial_{jk} \mathbf{G}_k + c_R \partial_{jk}^2 \mathbf{G}_k.
\end{aligned}$$

# Accuracy-Preserving Source Term Quadrature Formulas for Three-Dimension

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$$\sum_{k\in\{k_j\}} \frac{\mathbf{G}_L+\mathbf{G}_R}{2} V_{jk} = \sum_{k\in\{k_j\}} \left( \frac{a_L+a_R}{2}\mathbf{G}_j + \frac{a_R+b_L+b_R}{2}\partial_{jk}\mathbf{G}_j + \frac{2(b_R+c_R+c_L)+a_R}{4}\partial^2_{jk}\mathbf{G}_j \right) V_{jk}.$$

Two conditions are obtained by consistency and elimination of the first-order error terms:

$$\boxed{a_L + a_R = 2, \quad a_R + b_L + b_R = 0}$$

Compatibility condition on a regular tetrahedral grid

$$\frac{1}{V_j}\sum_{k\in\{k_j\}} \frac{\mathbf{G}_L+\mathbf{G}_R}{2} V_{jk} = \mathbf{G}_j + \frac{5h^2}{36}(2c_L+2b_R+2c_R+a_R)(\partial_{xx}\mathbf{G}_j+\partial_{yy}\mathbf{G}_j+\partial_{zz}\mathbf{G}_j-\partial_{xy}\mathbf{G}_j-\partial_{yz}\mathbf{G}_j+\partial_{zx}\mathbf{G}_j)$$

$$\mathcal{T}_j = \mathrm{div}\mathbf{F}^i - \mathbf{S}_j + \mathbf{G}_j - \frac{h^2}{12}\left[\partial_{xx}\mathbf{r}_j + \partial_{yy}\mathbf{r}_j + \partial_{zz}\mathbf{r}_j - \partial_{xy}\mathbf{r}_j - \partial_{yz}\mathbf{r}_j + \partial_{zx}\mathbf{r}_j\right] + O(h^3)$$

$$\boxed{a_R = -2(b_R + c_R + c_L) - \frac{3}{5}}$$

# Accuracy-Preserving Source Term Quadrature Formulas for Three-Dimension

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The quadrature formula achieves third-order accuracy on arbitrary tetrahedral grids if the coefficients satisfy the three conditions:

$$\boxed{a_L + a_R = 2, \quad a_R + b_L + b_R = 0, \quad a_R = -2(b_R + c_R + c_L) - \frac{3}{5}.}$$

| | Grid type | $a_L$ | $b_L$ | $c_L$ | $a_R$ | $b_R$ | $c_R$ |
|---|---|---|---|---|---|---|---|
| Regular | Regular Tetrahedra | $13/5$ | $0$ | $0$ | $-3/5$ | $0$ | $0$ |
| Compact | Arbitrary Tetrahedra | $13/5$ | $3/5$ | $0$ | $-3/5$ | $0$ | $0$ |
| Economical(1) | Arbitrary Tetrahedra | $1$ | $-1/5$ | $0$ | $1$ | $-4/5$ | $0$ |
| One-sided | Arbitrary Tetrahedra | $2$ | $0$ | -3/10 | $0$ | $0$ | $0$ |
| Symmetric | Arbitrary Tetrahedra | $1$ | $-1/2$ | $-3/20$ | $1$ | $-1/2$ | $-3/20$ |

These economical formulas bring a substantial saving in computational cost in 3D since there is no need to compute and store second derivatives

Implementation of the formulas in FUN3D:

$$\nabla \mathbf{G} \quad = \quad \alpha \nabla \mathbf{U} + \sum_{k=1}^{m} \alpha_{n-(k-1)} \nabla \mathbf{U}^{n-(k-1)}$$

- Compact formula, most efficient for unsteady computations
- FUN3D-i3rd needs to store the solution gradients at the previous time steps.
- HNS does not require additional work nor storage to evaluate $\nabla G$.

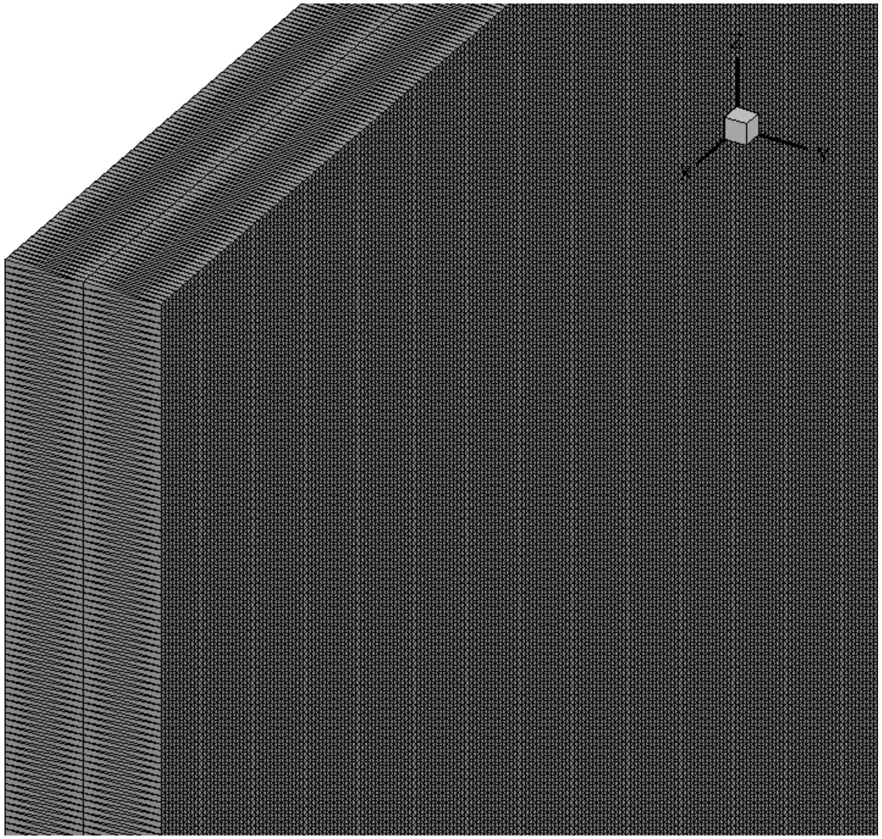
$$V_j \frac{d\mathbf{U}_j}{d\tau} = -\mathbf{P}_j \left( \sum_{k \in \{k_j\}} \mathbf{\Phi}_{jk} A_{jk} + \frac{1}{2} (\mathbf{G}_L + \mathbf{G}_R) V_{jk} - \mathbf{S}_j V_j \right),$$

## Numerical Results:

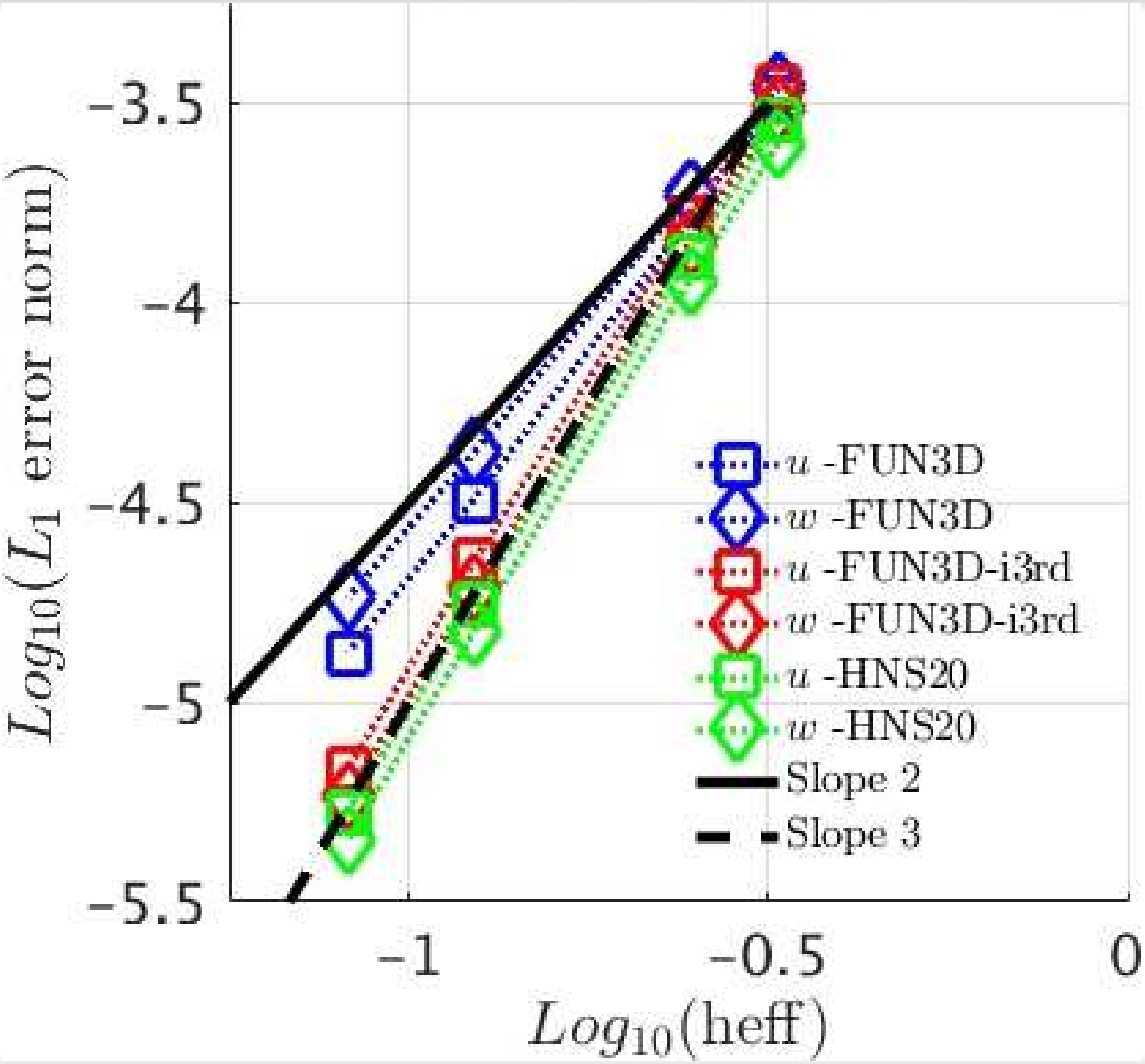
– Accuracy verification with inviscid moving vortex
– Unsteady viscous flow pass a cylinder

# Accuracy Verification with Moving Vortex
# Regular tetrahedral mesh

- Exact solution is an inviscid moving vortex with y-velocity set to zero
- Ma=0.5, Rey = 1million
- 3D rectangular domain x = [-20,20], z = [-10,10]
- Vortex moving along z = 0 line starting at location of x=-10
- BDF 3 with $\Delta t = 0.02$
- A series of regular tetrahedral meshes have been generated:
  - 241x121x3, 321x161x3, 641x321x3, 961x481x3



# Accuracy Verification with Moving Vortex
## Regular tetrahedral mesh



# Moving Vortex Contours
# Regular tetrahedral mesh

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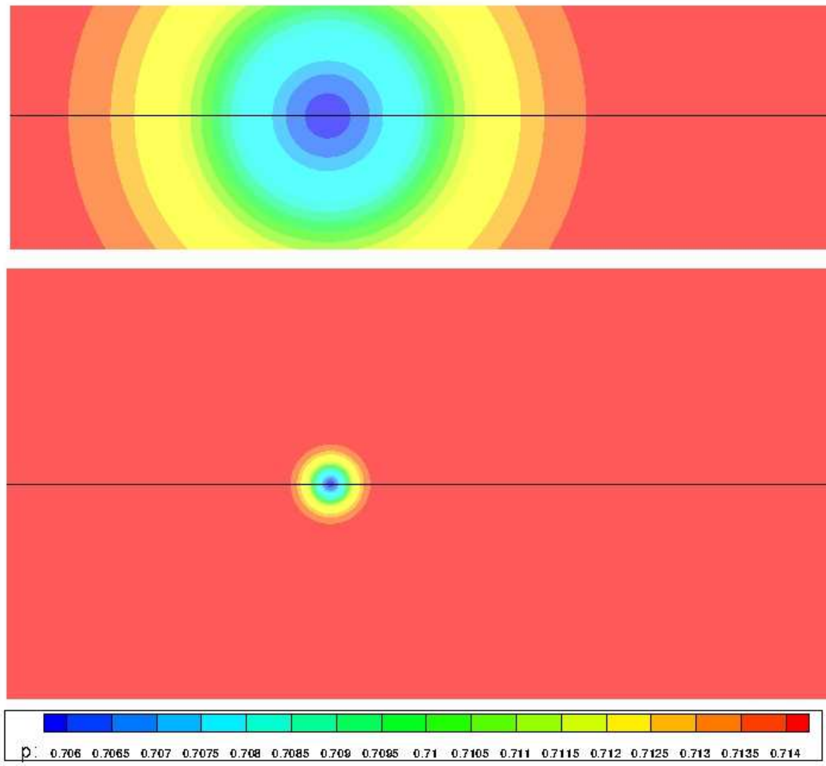
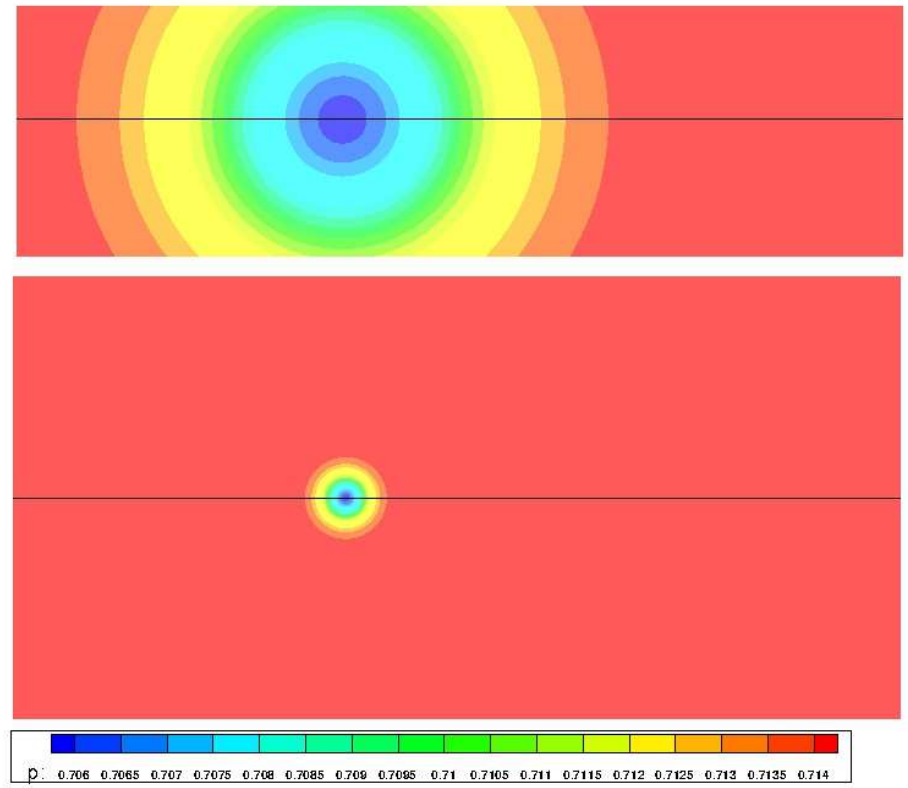


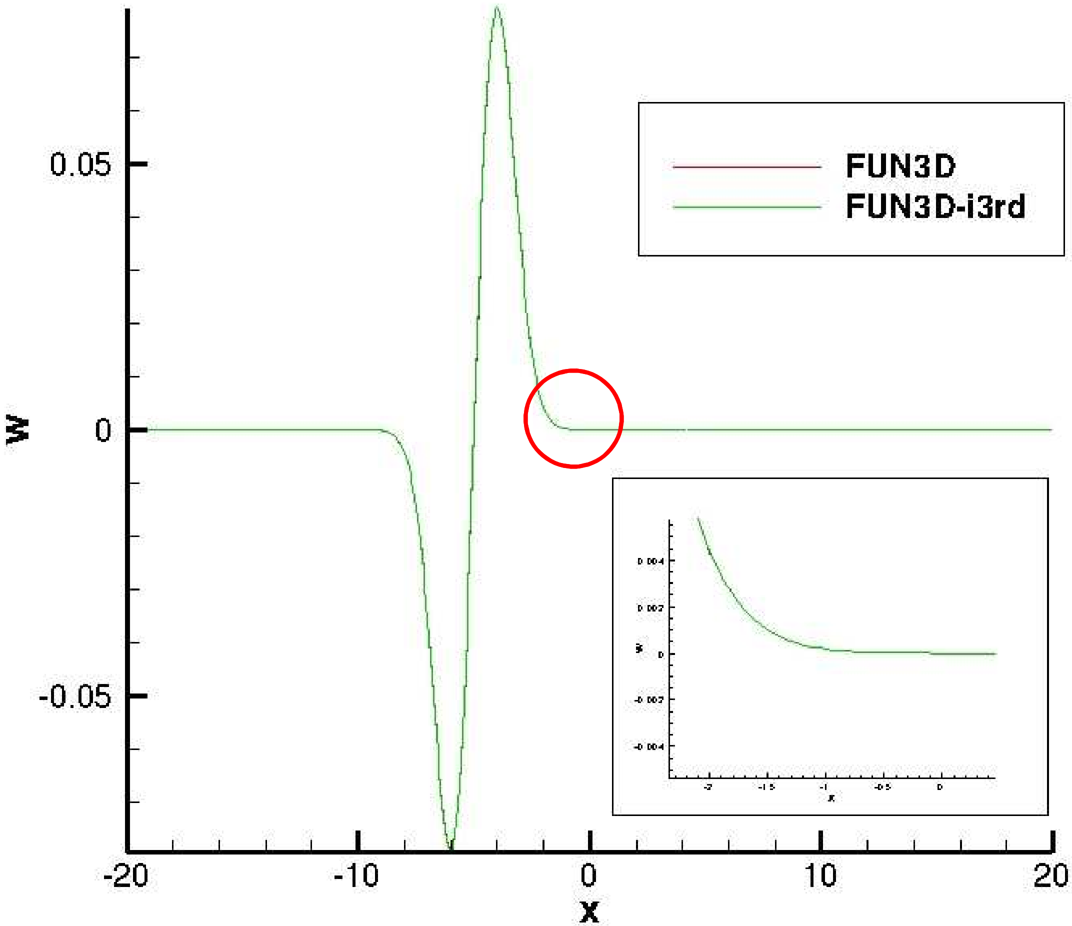

FUN3D




FUN3D-i3rd

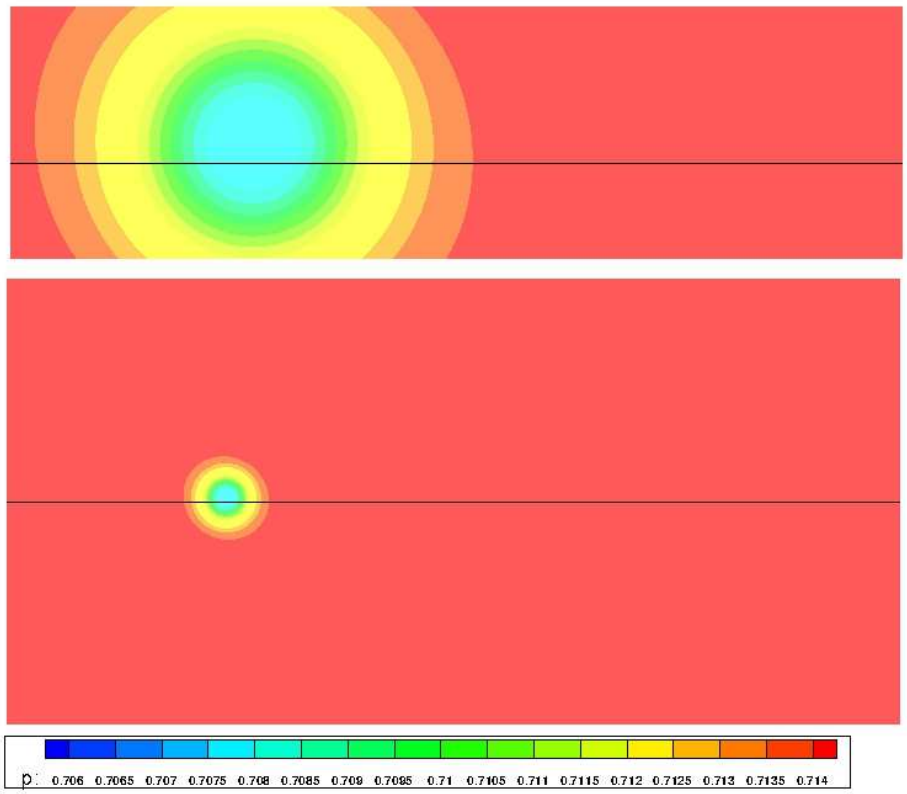
T = 10

# Moving Vortex Contours
# Regular tetrahedral mesh

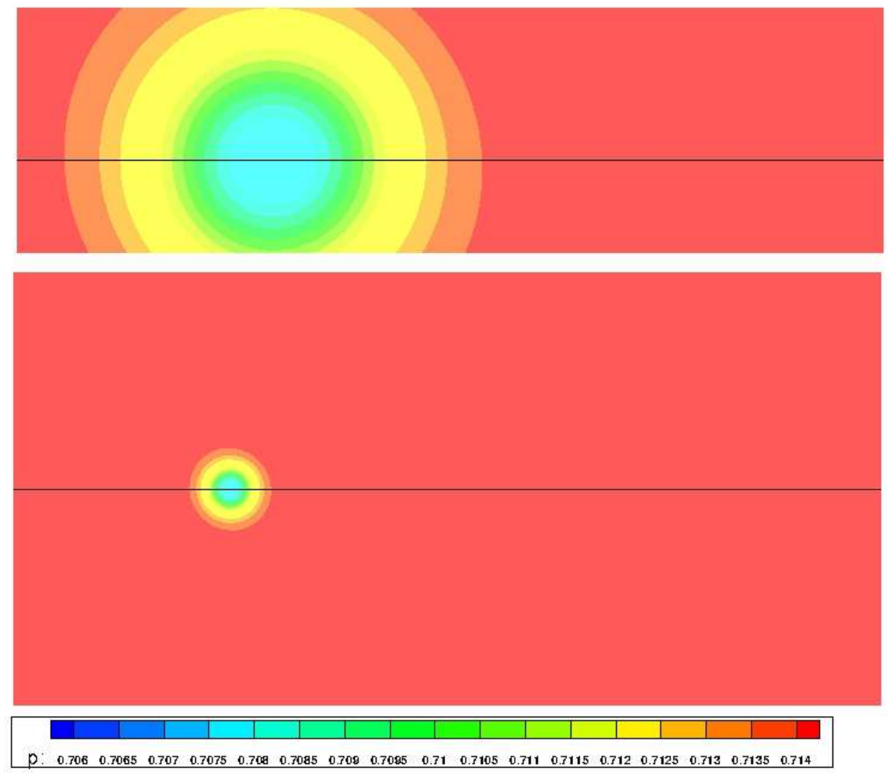



T = 10

# Moving Vortex Contours
# Regular tetrahedral mesh



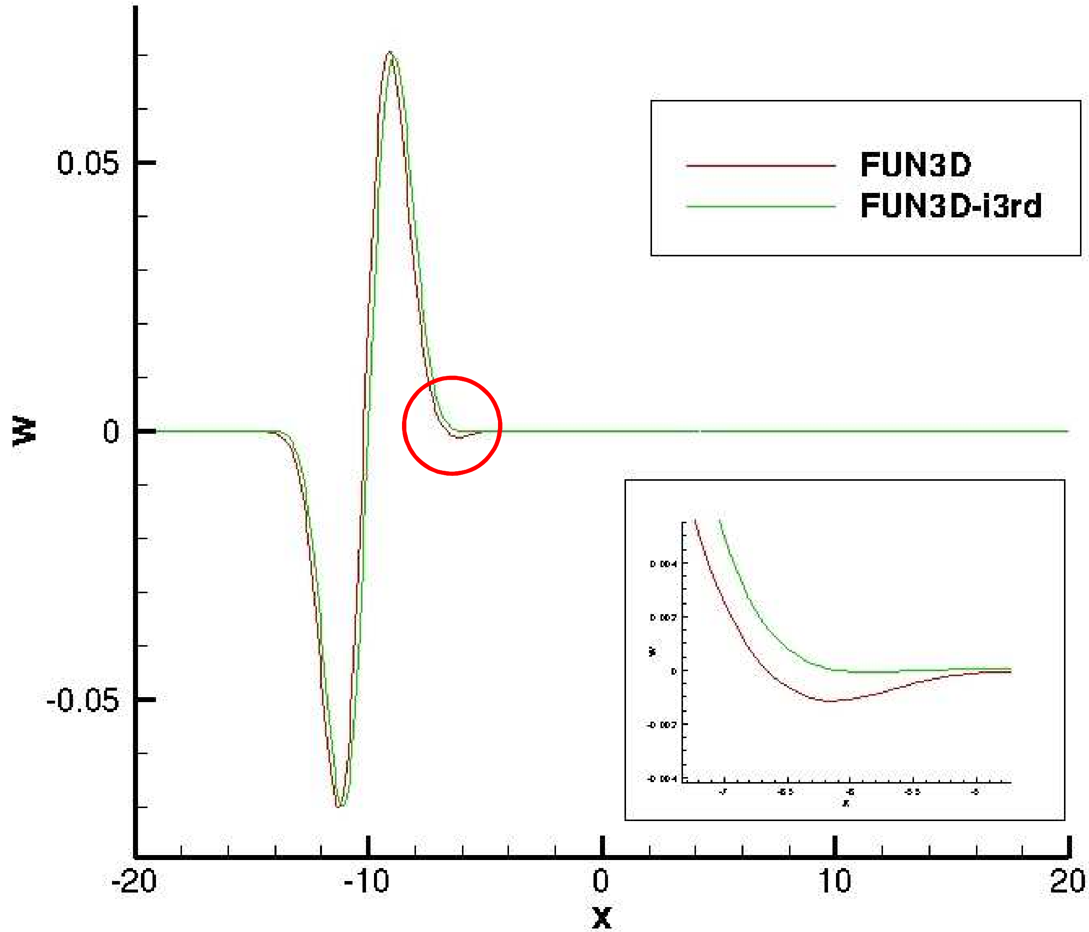

FUN3D




FUN3D-i3rd

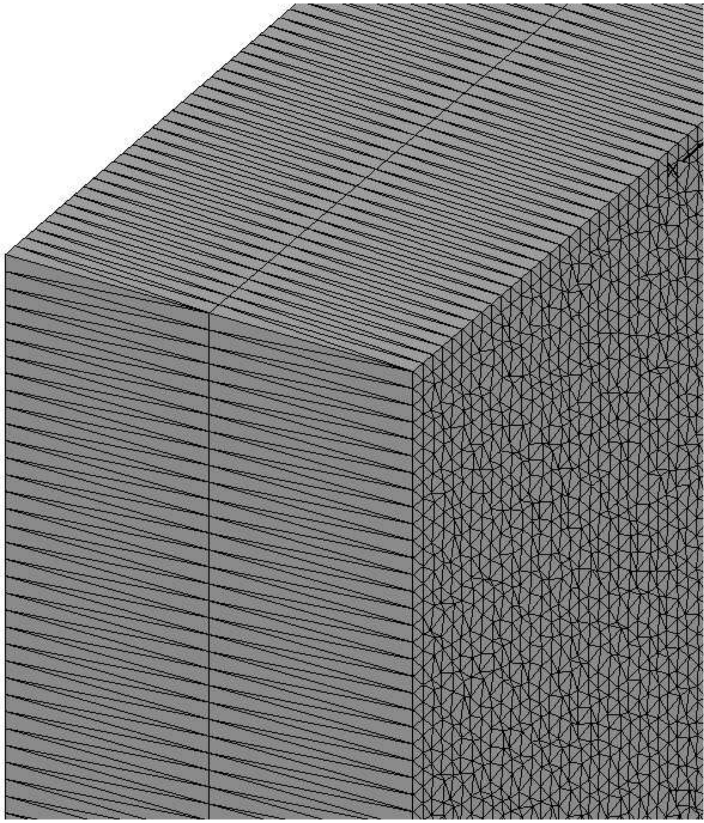
T = 800 (10 cycles)

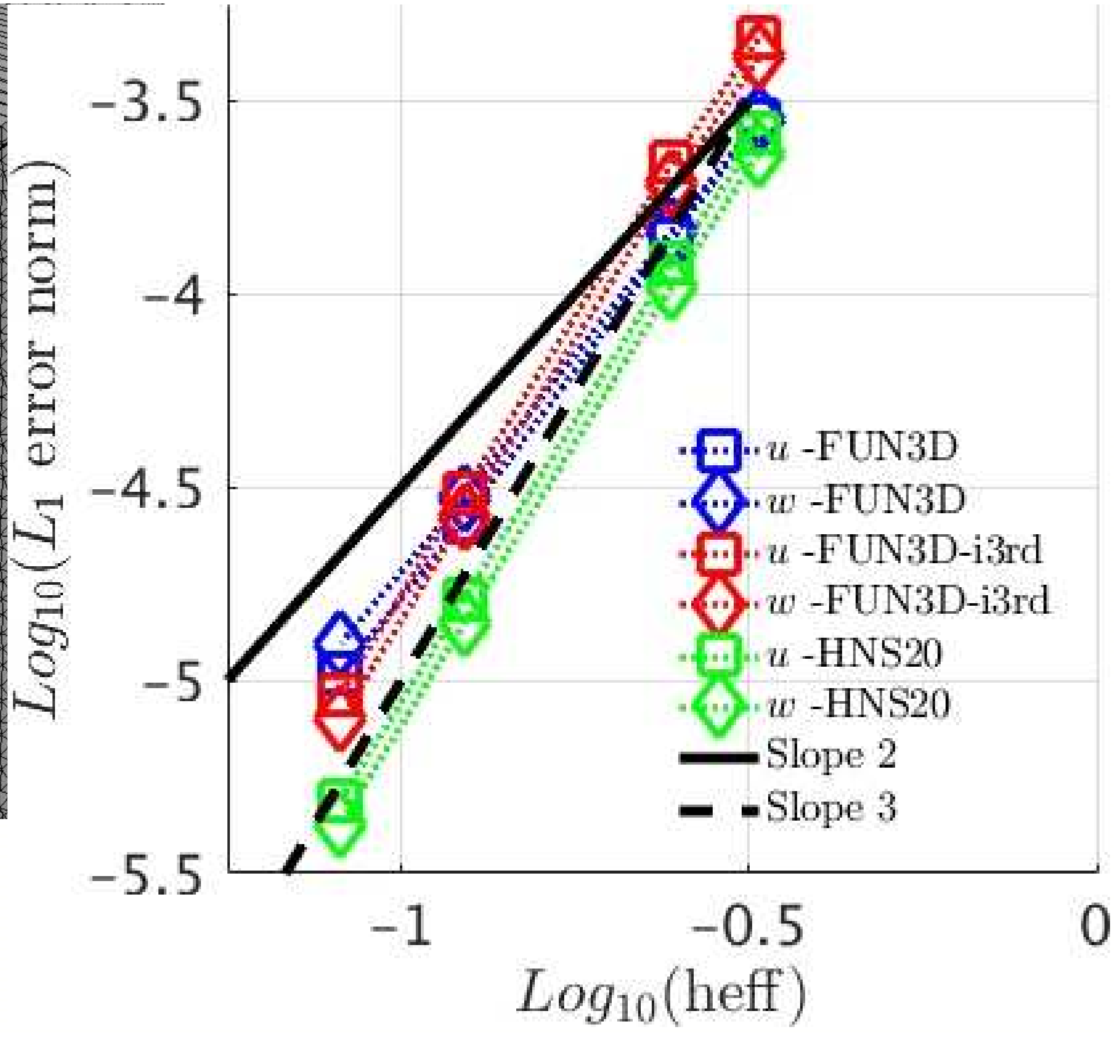
# Moving Vortex Contours
Regular tetrahedral mesh



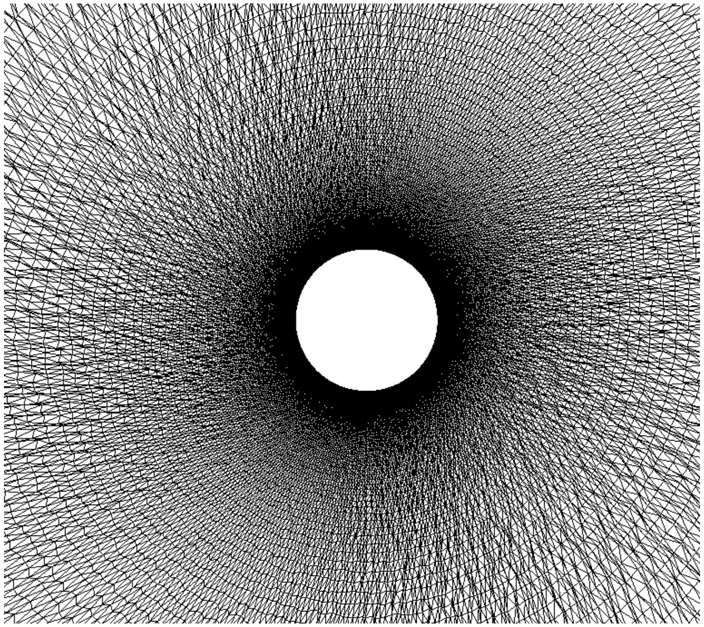

T = 800
(10 cycles)

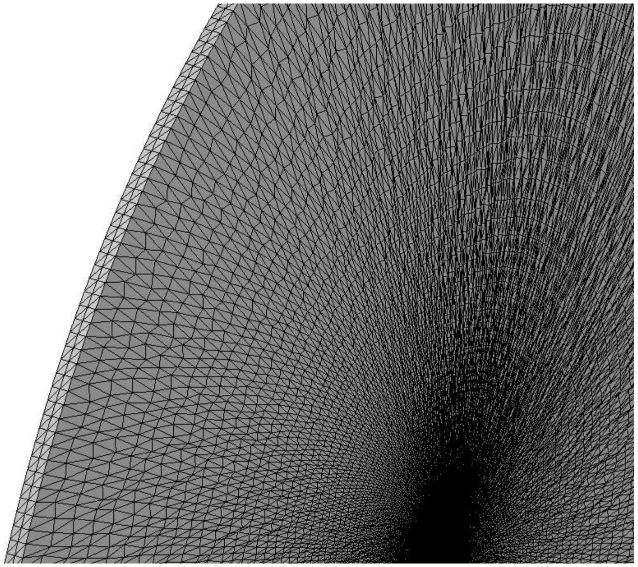
# Accuracy Verification with Moving Vortex
## Irregular tetrahedral mesh





# Vortex Shedding from Cylinder

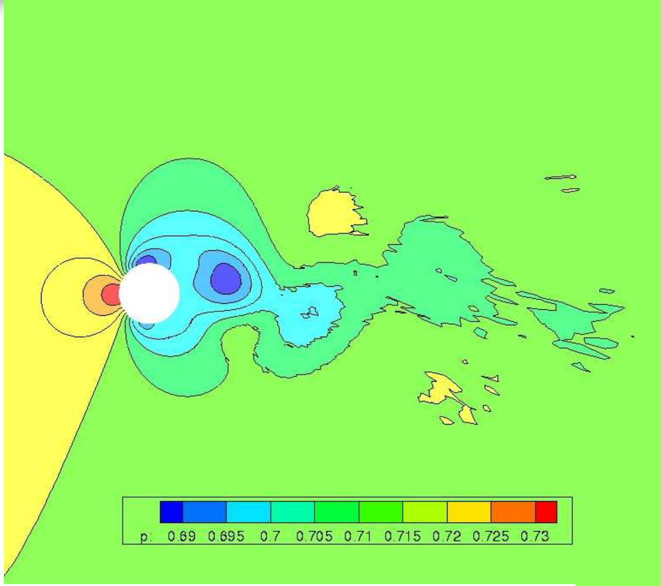




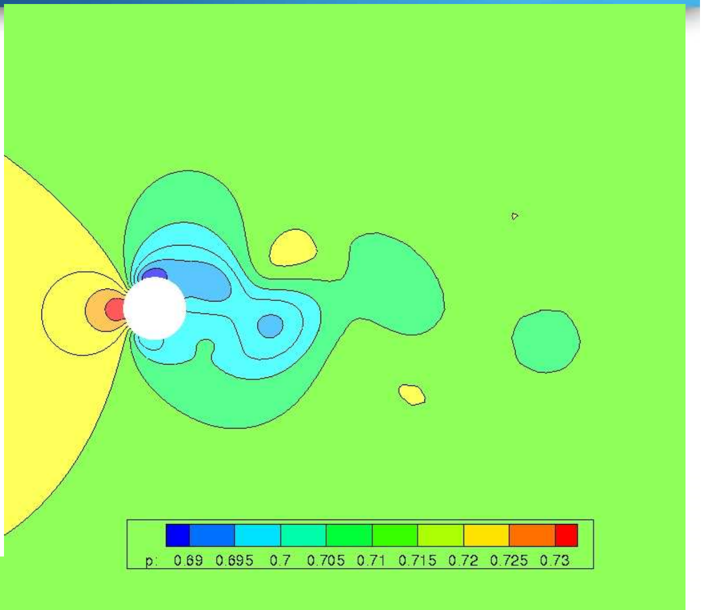
Mach = 0.2
Re = 150

Mesh size:
 401 x 201

# Vortex Shedding from Cylinder

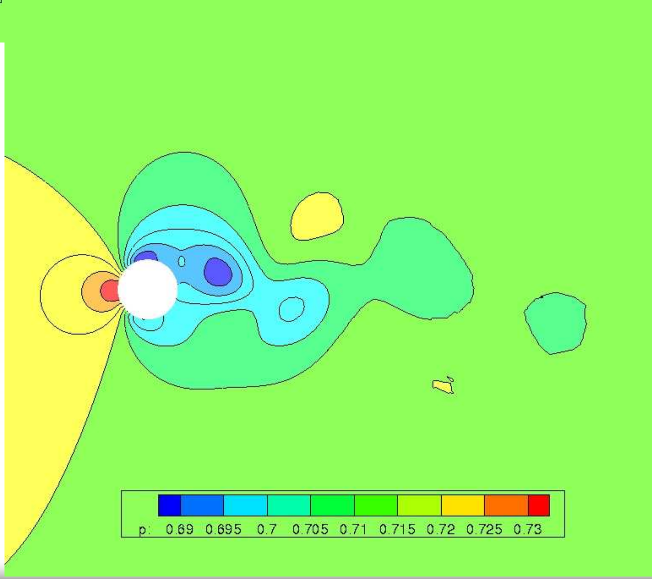



FUN3D




FUN3D-i3rd

Pressure Contour close to Cylinder at solution time 300




HNS20

# Vortex Shedding from Cylinder

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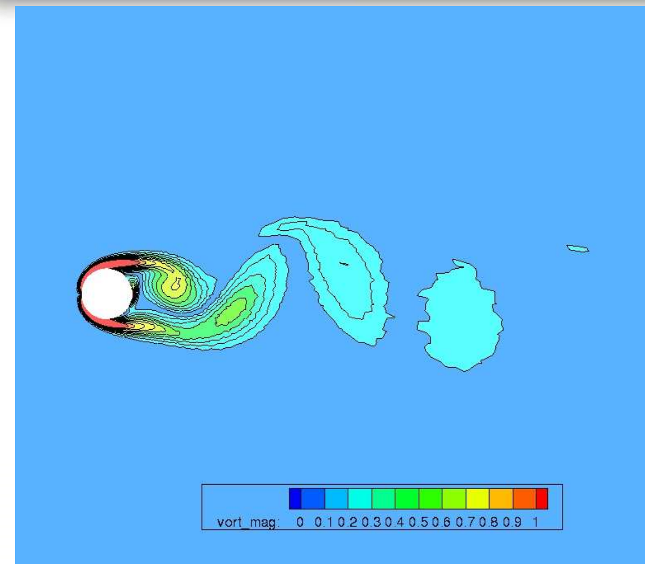
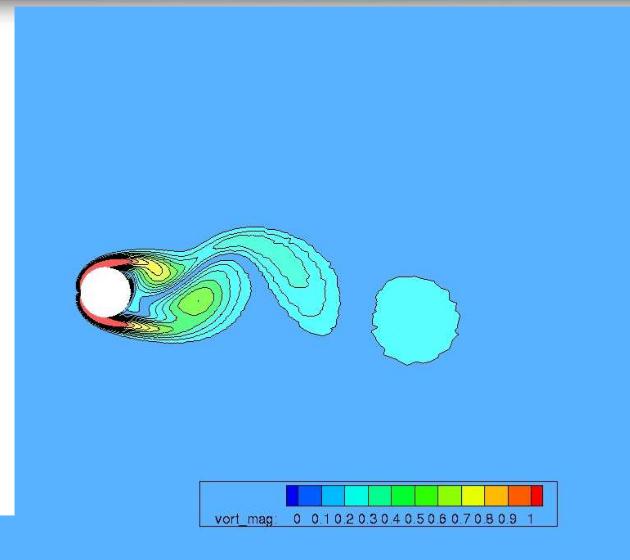
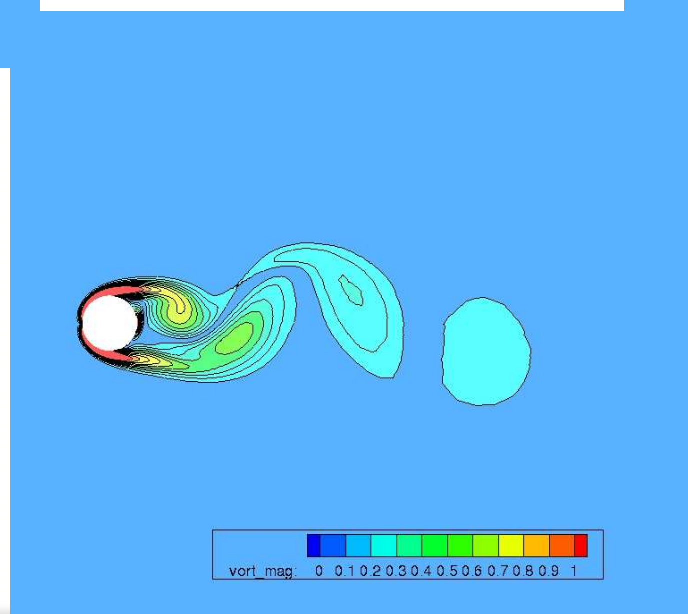



FUN3D




FUN3D-i3rd

Vorticity Magnitude Contour close to Cylinder at solution time 300




HNS20

# Conclusions

- Developed third-order edge based schemes for inviscid unsteady problems where the physical time derivatives are discretized by BDF in time and incorporated as source terms.
- Developed new accuracy-preserving source term quadrature formulas which does not require second derivatives at all.
- Two approaches are implemented into FUN3D to achieve third-order accuracy in inviscid limits.
  - FUN3D-i3rd, applying third-order edge based scheme to the inviscid terms with a quadratic LSQ fit with Galerkin viscous scheme.
  - HNS20, the hyperbolic Navier-Stokes method.

# Future Work

- Extend HNS20 to turbulent flow
- Develop a more smart quadratic LSQ fit method
- Achieve full third-order accuracy in inviscid and viscous terms with HNS20
- Make the HNS20 method more efficient in Navier-stokes computations

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