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Two Ways to Extend Diffusion Schemes to Navier-Stokes Schemes: Gradient Formula or Upwind Flux

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In this paper, we extend the diffusion schemes derived in the AIAA paper 2010-5093 to the Navier-Stokes schemes. There are two ways to do it. One is a widely-used approach: apply the gradient formula identified in the diffusion scheme to the gradients in the physical viscous flux. The other is a new way. It is a direct extension of the hyperbolic approach to the Navier-Stokes equations: a viscous scheme is derived from an upwind scheme applied to a hyperbolic model for the viscous term. Viscous discretizations derived from the two approaches are presented for the finite-volume method. A particular emphasis is given on the damping term which is essential to robust and accurate viscous computations. It is demonstrated that the hyperbolic approach is a general approach by which the damping term is automatically introduced into the viscous discretization. Preliminary results show that both approaches yield Navier-Stokes schemes of comparable accuracy and the lack of damping leads to inaccurate solutions.

1. Introduction

Towards highly efficient, robust, and accurate viscous simulations, considerable effort has been devoted to the development of diffusion schemes particularly in high-order methods [1,2,3,4,5,6,7,8] and unstructured grid methods [9,10,11,12,13,14,15,16]. One of the fundamental difficulties in developing successful diffusion schemes is the lack of general guiding principle. Unlike the advection equation (or hyperbolic systems in general) for which 'upwinding' has been found a useful guiding principle, in one form or another, for developing successful advection schemes, the isotropic nature of the diffusion equation does not serve by itself as a practical guide for developing successful diffusion schemes. To overcome this difficulty, we proposed one possible guiding principle for diffusion in Ref. [9]. It is to discretize a hyperbolic model for diffusion by an advection scheme, and derive a numerical scheme for the diffusion equation from the result. The system to be discretized being hyperbolic, the principle of 'upwinding' is now directly applicable to diffusion. Its general applicability has been demonstrated in Refs. [9, 17] for node/cell-centered finite-volume, residual-distribution, discontinuous Galerkin, and spectral-volume methods. In Ref. [9], we also identified two essential elements in robust and accurate diffusion schemes: *consistent and damping terms*. The former is responsible for approximating the physical diffusive flux consistently while the latter is for providing high-frequency damping. The impact of the damping term has been demonstrated for highly-skewed typical viscous grids in Refs. [9, 17]: a good amount of damping leads to remarkably smooth and accurate solutions on highly-skewed irregular grids; the lack of damping leads to extremely inaccurate solutions, instability, and, in some case, inconsistency also. A particularly useful feature of the general hyperbolic approach introduced in Ref. [9] is that the resulting diffusion schemes are automatically equipped with a damping term; it is inherited from the dissipation term of the generating advection scheme. Through the hyperbolic approach, the form of the damping term has been identified in various discretization methods in Refs. [9, 17], including the residual-distribution method.

In this paper, we extend the diffusion schemes derived in Refs. [9,17] to the Navier-Stoke schemes. There are two ways to do it. They are closely related to the two approaches to the construction of diffusion schemes: the gradient approach and the hyperbolic approach. The gradient approach is a widely-used approach. In this approach, the construction of a diffusion scheme boils down to the construction of the gradients (e.g., at the interface) by which the physical diffusive flux is directly evaluated. Many methods for diffusion belong to this category. The diffusion scheme derived from this approach can be readily extended to the viscous term: apply the gradient formula to the velocity and temperature gradients in the physical viscous flux. To employ this approach, we write our diffusion

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scheme in the interface-gradient form, and identify the corresponding gradient formula; it is then applied to evaluate the gradients in the physical viscous flux. On the other hand, it is also possible to directly extend the hyperbolic approach to the Navier-Stokes equations. To enable this extension, we utilize a hyperbolic system model for the viscous term introduced in Ref. [18]. Viscous discretization is then derived from an upwind scheme applied to the hyperbolic viscous system. Taking the finite-volume method as an example, we demonstrate that a high-frequency damping term is automatically introduced into the resulting viscous scheme; no special consideration is necessary to incorporate them. Preliminary results are presented to demonstrate the impact of the damping term on the solution accuracy. We emphasize that *this paper is not merely proposing a new viscous discretization but introducing a general hyperbolic approach to the viscous discretization.* Various other viscous discretizations can be constructed, for example, by applying other numerical fluxes to the hyperbolic viscous system. Also, the hyperbolic approach can be employed in various other discretization methods including the residual-distribution method. We point out also that this paper aims at generating time-accurate viscous schemes whereas the paper [18] aims at generating schemes for steady state computations (it can be made time-accurate by the dual-time stepping method). Although using a similar hyperbolic system, the viscous discretizations derived from the two methods will be very different. The schemes derived in this paper require no extra variables/equations to be stored/solved, and the explicit time step will be subject to the 'traditional' $O(h^2)$ restriction, not $O(h)$. The latter is a consequence of a striking difference in the definition of the relaxation times for the hyperbolic viscous system.

We begin this paper by describing the diffusion scheme introduced in Ref. [18], then extend it to the Naiver-Stokes equations in two different ways, subsequently present numerical results for a simple test problem, and finally conclude the paper with remarks.

2. Diffusion Schemes

2.1. One Dimension

Consider the diffusion equation in one dimension:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

where ν is a positive constant. To derive a numerical scheme for the diffusion equation, we introduced a general approach in Refs. [9,17] in which we begin by discretizing a first-order model for diffusion:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial p}{\partial x}, \quad \frac{\partial p}{\partial t} = \frac{1}{T_r}\left(\frac{\partial u}{\partial x} - p\right), \tag{2.2}$$

where p is a variable that approaches the solution gradient at the time scale of the relaxation time, $T_r(>0)$. Note that the system is not equivalent to the diffusion equation because p is not equal to the solution gradient except in a steady state or in the limit $T_r \to 0$. Hence, the second equation is the possible source of inconsistency between the two models. In the vector form, the system is written as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}, \tag{2.3}$$

where

$$\mathbf{U} = \begin{bmatrix} u \\ p \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} -\nu p \\ -u/T_r \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ -p/T_r \end{bmatrix}. \tag{2.4}$$

This system is hyperbolic. The Jacobian matrix, $\mathbf{A} = \partial \mathbf{F}/\partial \mathbf{U}$, has the following eigenvalues:

$$-\sqrt{\frac{\nu}{T_r}}, \quad \sqrt{\frac{\nu}{T_r}}, \tag{2.5}$$

which are real for any positive T_r, and the corresponding eigenvectors are linearly independent. It follows that the system describes a wave traveling in the opposite directions at the same speed. To derive a diffusion scheme, we first discretize the hyperbolic system. Here, we consider the finite-volume method. On a one-dimensional grid of N nodes with uniform spacing, h, with the solution data stored at the nodes denoted by x_j, $j = 1, 2, 3, \ldots, N$, we consider the following semi-discrete finite-volume discretization over the dual volume $I_j = [x_{j-1/2}, x_{j+1/2}]$:

$$\frac{d\mathbf{U}_j}{dt} = -\frac{1}{h}\left[\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right] + \frac{1}{h}\int_{I_j} \mathbf{S}\, dx, \tag{2.6}$$

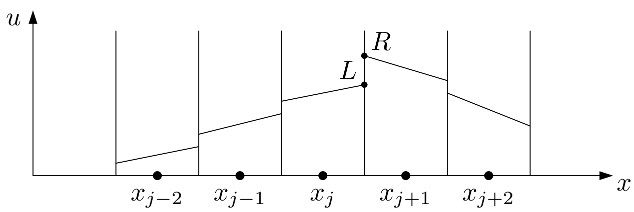


Figure 2.1. Discontinuous piecewise linear data in one dimension.

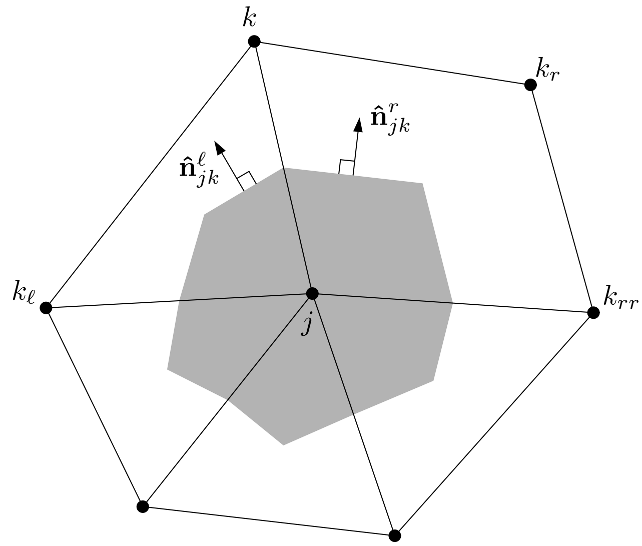


Figure 2.2. Dual control volume for node-centered finite-volume schemes with unit normals associated with an edge, $\{j, k\}$.

where $\mathbf{U}_j$ is the volume-averaged solution at node j. Note that the source term is irrelevant because it has a nonzero entry only in the second equation which we will ignore after the discretization. The interface flux, $\mathbf{F}_{j+1/2}$, is defined by the upwind flux:

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left[\mathbf{F}_R + \mathbf{F}_L\right] - \frac{1}{2}\left|\mathbf{A}\right|(\mathbf{U}_R - \mathbf{U}_L) = \frac{1}{2}\left[\mathbf{F}_R + \mathbf{F}_L\right] - \frac{1}{2}\sqrt{\frac{\nu}{T_r}}(\mathbf{U}_R - \mathbf{U}_L), \tag{2.7}$$

where $\mathbf{F}_L$ and $\mathbf{F}_R$ denote the physical flux evaluated by the left and right states, $\mathbf{U}_L$ and $\mathbf{U}_R$, respectively. For second-order accuracy, we define a reconstructed piecewise linear solution within each dual volume, and extrapolate the left and right states to the interface (see Figure 2.1). The gradient reconstruction is performed in each control volume by the central-difference formula.

As discussed in Refs. [9, 17], the relaxation time is chosen to keep the hyperbolic behavior of the system over every explicit time step:

$$T_r = \frac{h^2}{\alpha^2 \nu}, \tag{2.8}$$

where α is a parameter of $O(1)$, thus yielding

$$\mathbf{F}_{j+1/2} \quad = \quad \frac{1}{2}\left[\mathbf{F}_R + \mathbf{F}_L\right] - \frac{\nu\alpha}{2h}(\mathbf{U}_R - \mathbf{U}_L). \tag{2.9}$$

Note that this upwind scheme is not necessarily consistent with the diffusion equation because the second variable, p_j, may not be an accurate approximation to the solution gradient. We now derive a diffusion scheme by ignoring the second equation. The result is

$$\frac{du_j}{dt} = -\frac{1}{h}\left[f_{j+1/2} - f_{j-1/2}\right], \tag{2.10}$$

where

$$f_{j+1/2} = -\frac{\nu}{2}\left[p_R + p_L\right] - \frac{\nu\alpha}{2h}(u_R - u_L). \tag{2.11}$$

We evaluate the gradients, p_L and p_R, (whose evolution equation has just been ignored) consistently by differentiating the numerical solution on the left and right control-volumes respectively, leading to the central-difference formula:

$$p_L = \left(\frac{\partial u}{\partial x}\right)_j = \frac{u_{j+1} - u_{j-1}}{2h}, \quad p_R = \left(\frac{\partial u}{\partial x}\right)_{j+1} = \frac{u_{j+2} - u_j}{2h}. \tag{2.12}$$

Note that these gradients are not interface gradients but the nodal gradients defined at the center of the control-volume. Note also that the left and right solutions, u_L and u_R, are the extrapolated interface values:

$$u_L = u_j + \frac{h}{2}\left(\frac{\partial u}{\partial x}\right)_j, \quad u_R = u_{j+1} - \frac{h}{2}\left(\frac{\partial u}{\partial x}\right)_{j+1}, \tag{2.13}$$

The resulting scheme is a consistent time-accurate diffusion scheme, i.e., a scheme for the diffusion equation (2.1) because the second equation, which is the source of inconsistency between the two models, has been ignored and p is now evaluated consistently with the solution gradient. As discussed in details in Refs. [9,17], this scheme reduces to the three-point central-difference scheme for $\alpha = 2$ and becomes fourth-order accurate for $\alpha = 8/3$. It is important to note that the first term in the numerical flux (2.11) is called the *consistent term* which approximates the physical flux consistently while the second term is a quantity of $O(h^2)$ and called the *damping term*. As shown in Refs. [9,17], the damping term plays a role of high-frequency damping; α is the parameter that controls the amount of the damping. This hyperbolic approach is a general approach applicable to various discretization methods as demonstrated in Refs. [9, 17] for the node/cell-centered finite-volume, residual-distribution, discontinuous Galerkin, and spectral-volume methods. One of the advantages of this particular approach is that the damping term is automatically incorporated via the dissipation term of the upwind flux, which otherwise requires a careful consideration. This approach can be extended to the Navier-Stokes equations provided a suitable hyperbolic system is available.

On the other hand, the derived diffusion scheme (2.10) can be written in the interface-gradient form:

$$\frac{du_j}{dt} = \frac{\nu}{h}\left[\left(\frac{\partial u}{\partial x}\right)_{j+1/2} - \left(\frac{\partial u}{\partial x}\right)_{j-1/2}\right], \tag{2.14}$$

where

$$\left(\frac{\partial u}{\partial x}\right)_{j+1/2} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)_j + \left(\frac{\partial u}{\partial x}\right)_{j+1}\right] + \frac{\alpha}{2h}\left(u_R - u_L\right), \tag{2.15}$$

and similarly for the other interface. Equation (2.15) defines a one-parameter-family formula for the interface gradient which leads to the diffusion scheme equivalent to the three-point central-difference scheme for $\alpha = 2$ and the fourth-order scheme for $\alpha = 8/3$. This particular formulation allows a simple extension to the Navier-Stokes schemes: directly evaluate the gradients in the viscous flux by the gradient formula. This is a widely-used approach. Although it is very simple, a robust gradient formula that incorporates a damping mechanism must be available in this approach. The above formula is one such example.

2.2. Two Dimensions

Consider the diffusion equation in two dimensions:

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{2.16}$$

It is straightforward to apply the same hyperbolic approach to the two-dimensional equation [9, 17]. In the edge-based finite-volume method, it yields the following diffusion scheme:

$$\frac{du_j}{dt} = \frac{1}{V_j}\sum_{k\in\{K_j\}} \phi_{jk} A_{jk}, \tag{2.17}$$

where u_j is the volume-averaged solution at the node j, V_j denotes the volume of the dual control volume around j, $\{K_j\}$ is a set of neighbors of j, and A_{jk} is the magnitude of the directed area vector, $\mathbf{n}_{jk} = \mathbf{n}^{\ell}_{jk} + \mathbf{n}^{r}_{jk}$ (see Figure 2.2). In each dual control volume, we reconstruct the solution gradient (e.g., by the least-squares reconstruction) and define a linear variation in the solution. The interface flux, ϕ_{jk}, defined at the midpoint of the edge, which was derived from an upwind flux [9, 17], is given by

$$\phi_{jk} = \frac{\nu}{2}\left[(\nabla u)_j + (\nabla u)_k\right]\cdot\hat{\mathbf{n}}_{jk} + \frac{1}{2}\sqrt{\frac{\nu}{T_r}}\left(u_R - u_L\right), \tag{2.18}$$

where $(\nabla u)_j$ and $(\nabla u)_k$ denote the reconstructed gradients at the nodes, j and k, respectively, u_R and u_L are the extrapolated solution values at the edge midpoint from the nodes j and k, respectively, and $\hat{\mathbf{n}}_{jk}$ is the unit directed area vector. The relaxation time T_r is defined by

$$T_r = \frac{L_r^2}{\alpha^2 \nu}, \quad L_r = \frac{1}{2}\,\Delta L_{jk}\,|\hat{\mathbf{e}}_{jk} \cdot \hat{\mathbf{n}}_{jk}|, \tag{2.19}$$

where $\hat{\mathbf{e}}_{jk}$ denotes the unit vector along the edge, and ΔL_{jk} is the length of the edge. A complete derivation can be found in Refs. [9,17]. As in one dimension, the first term in the interface flux is the consistent term and the second term is the damping term. As demonstrated in Refs. [9,17], the damping term is essential to robust and accurate computations on highly-skewed grids. The skewness measure, $\hat{\mathbf{e}}_{jk} \cdot \hat{\mathbf{n}}_{jk}$, has an important effect of amplifying the damping at skewed faces. On a structured mesh, the scheme reduces to the central-difference scheme for $\alpha = 1$ and achieves fourth-order accuracy for $\alpha = 4/3$ [9,17]. The choice $\alpha = 4/3$ gives highly accurate solutions on highly-skewed irregular grids although not fourth-order accurate (see Refs. [9,17]). As in one dimension, the hyperbolic approach can be extended to the Navier-Stokes equations provided a suitable hyperbolic system is available in two dimensions.

It is possible to cast the diffusion scheme (2.18) in the interface-gradient form:

$$\phi_{jk} = \nu\, \nabla u|_{jk} \cdot \hat{\mathbf{n}}_{jk}, \tag{2.20}$$

where the interface gradient, $\nabla u|_{jk}$, is given by

$$\nabla u|_{jk} = \overline{\nabla u}_{jk} + \frac{\alpha}{2L_r}\,(u_R - u_L)\,\hat{\mathbf{n}}_{jk}, \quad \overline{\nabla u}_{jk} = \frac{1}{2}\left[(\nabla u)_j + (\nabla u)_k\right]. \tag{2.21}$$

This defines a one-parameter-family formula for the interface gradient which is equipped with a damping term. This formulation allows a simple and widely-used extension to the Navier-Stokes schemes: directly evaluate the gradients in the viscous flux by the above formula. Although it enables a very simple extension to the Navier-Stokes schemes, this approach requires a well-designed gradient formula that incorporates a mechanism to introduce the damping effect in the resulting scheme. For unstructured finite-volume methods, some successful formulas have been developed in the past [19,20,21], but the distinction between the consistent and damping terms did not seem well recognized. Consequently, those gradient formulas are designed as consistent gradient approximations; for robustness, the edge-term, $(u_k - u_j)/\Delta L_{jk}$, is incorporated in a consistent manner. The gradient formula (2.21) is a general and flexible formula containing such well-known formulas as special cases [9,17]. The edge-term is incorporated, in effect, in the damping term in the gradient formula (2.21) [9,17].

3. Extensions to the Navier-Stokes Equations

We will now extend the diffusion scheme to the Navier-Stokes equations. As implied by the discussion in the previous section, there are two ways to do it. One is to apply the gradient formula to the velocity and temperature gradients in the physical viscous flux. The other is to derive a numerical viscous flux from an upwind flux applied to a hyperbolic model for the viscous term (i.e., a direct extension of the approach). From here on, u and p denote the x-component of the velocity and the pressure, respectively.

3.1. One Dimension

3.1.1. Navier-Stokes Equations in One Dimension

Consider the compressible Navier-Stokes equations in one dimension:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{0}, \tag{3.1}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, \quad \mathbf{F} = \mathbf{F}^{inv} + \mathbf{F}^{vis} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{bmatrix} + \begin{bmatrix} 0 \\ -\tau \\ -\tau u + q \end{bmatrix}, \tag{3.2}$$

where $\mathbf{F}^{inv}$ and $\mathbf{F}^{vis}$ denote respectively the inviscid and viscous fluxes, ρ is the density, u is the velocity, p is the pressure, E is the specific total energy, and $H = E + p/\rho$ is the specific total enthalpy. The viscous stress τ and the heat flux q are given by

$$\tau = \frac{4}{3}\mu\frac{\partial u}{\partial x}, \quad q = -\frac{\mu}{Pr(\gamma-1)}\frac{\partial T}{\partial x}, \tag{3.3}$$

where γ is the ratio of specific heats, Pr is the Prandtl number, and μ is the viscosity given by Sutherland's law. All the quantities are assumed to be nondimensionalized by their free stream values except that the velocity and the pressure are scaled by the speed of sound a_∞ and the dynamic pressure $\rho_\infty a_\infty^2$, respectively. Then, the viscosity is given by the following scaled form of Sutherland's law:

$$\mu(T) = \frac{M_\infty}{Re_\infty}\frac{1 + C/T_\infty}{T + C/T_\infty}T^{\frac{3}{2}}, \tag{3.4}$$

where T is the nondimensional temperature, T_∞ is the dimensional free stream temperature, and $C = 110.5$ [K] is the Sutherland constant. The ratio of the free stream Mach number, M_∞, to the free stream Reynolds number, Re_∞, arise from the nondimensionalization. The system is closed by the nondimensionalized equation of state for ideal gases: $\gamma p = \rho T$.

3.1.2. *Finite-Volume Discretization*

To discretize the Navier-Stokes equations, we consider the finite-volume method. Integrating the Navier-Stokes system over a dual cell, $I_j = [x_{j-1/2}, x_{j+1/2}]$, we obtain

$$\frac{d\mathbf{U}_j}{dt} = -\frac{1}{h}\left[\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right], \tag{3.5}$$

where the interface flux consists of the inviscid and viscous parts:

$$\mathbf{F}_{j+1/2} = \mathbf{F}^{inv}_{j+1/2} + \mathbf{F}^{vis}_{j+1/2}. \tag{3.6}$$

For the inviscid part, we employ the upwind flux based on Roe's approximate Riemann solver [22]:

$$\mathbf{F}^{inv}_{j+1/2} = \frac{1}{2}\left[\mathbf{F}^{inv}(\mathbf{U}_L) + \mathbf{F}^{inv}(\mathbf{U}_R)\right] - \frac{1}{2}\Delta\mathbf{F}^{inv}, \tag{3.7}$$

where $\Delta\mathbf{F}^{inv} = |\hat{\mathbf{A}}^{inv}|(\mathbf{U}_R - \mathbf{U}_L)$ is the dissipation term, and $\hat{\mathbf{A}}^{inv}$ is the inviscid Jacobian evaluated by the Roe-averaged states. For second-order accuracy, we reconstruct a piecewise linear solution within each control volume, and the left and right states, $\mathbf{U}_L$ and $\mathbf{U}_R$, are extrapolated to the interface (see Figure 2.1). The reconstruction is performed in the primitive variables, (ρ, u, p). For time integration, we employ the forward Euler time-stepping scheme in this study. The viscous flux remains to be defined. We construct the viscous flux by extending the diffusion scheme in the previous section. There are two ways to do it.

3.1.3. *Viscous Flux via Gradient Formula*

We may directly evaluate the physical viscous flux at the interface:

$$\mathbf{F}^{vis}_{j+1/2} = \mathbf{F}^{vis}(\mathbf{U}_{j+1/2}, \nabla\mathbf{U}_{j+1/2}) = \begin{bmatrix} 0 \\ -\tau_{j+1/2} \\ -\tau_{j+1/2}\, u_{j+1/2} + q_{j+1/2} \end{bmatrix}, \tag{3.8}$$

where

$$\tau_{j+1/2} = \frac{4}{3}\mu_{j+1/2}\left(\frac{\partial u}{\partial x}\right)_{j+1/2}, \quad q_{j+1/2} = -\frac{\mu_{j+1/2}}{Pr(\gamma-1)}\left(\frac{\partial T}{\partial x}\right)_{j+1/2}. \tag{3.9}$$

At the interface, given the left and right states, (ρ_L, u_L, p_L) and (ρ_R, u_R, p_R), we compute the interface velocity and the viscosity by the arithmetic averages:

$$u_{j+1/2} = \frac{u_L + u_R}{2}, \quad \mu_{j+1/2} = \mu\left(\frac{T_L + T_R}{2}\right), \tag{3.10}$$

where $T_L = \gamma p_L/\rho_L$ and $T_R = \gamma p_R/\rho_R$. We then evaluate the velocity and temperature gradients at the interface by the gradient formula (2.15):

$$\left(\frac{\partial u}{\partial x}\right)_{j+1/2} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)_j + \left(\frac{\partial u}{\partial x}\right)_{j+1}\right] + \frac{\alpha}{2h}\left(u_R - u_L\right), \tag{3.11}$$

$$\left(\frac{\partial T}{\partial x}\right)_{j+1/2} = \frac{1}{\rho_{j+1/2}}\left[\gamma\left(\frac{\partial p}{\partial x}\right)_{j+1/2} - a_{j+1/2}^2\left(\frac{\partial \rho}{\partial x}\right)_{j+1/2}\right], \tag{3.12}$$

where $\rho_{j+1/2}$ and $a_{j+1/2}$ are the density and the speed of sound at the interface computed by the arithmetic averages of the density and the pressure, and

$$\left(\frac{\partial \rho}{\partial x}\right)_{j+1/2} = \frac{1}{2}\left[\left(\frac{\partial \rho}{\partial x}\right)_j + \left(\frac{\partial \rho}{\partial x}\right)_{j+1}\right] + \frac{\alpha}{2h}\left(\rho_R - \rho_L\right), \tag{3.13}$$

$$\left(\frac{\partial p}{\partial x}\right)_{j+1/2} = \frac{1}{2}\left[\left(\frac{\partial p}{\partial x}\right)_j + \left(\frac{\partial p}{\partial x}\right)_{j+1}\right] + \frac{\alpha}{2h}\left(p_R - p_L\right). \tag{3.14}$$

This completes the construction of the numerical viscous flux. This approach is a widely-used method for extending a diffusion scheme to the viscous scheme; but the application of the gradient formula (2.15) is new. It is emphasized that this approach generally requires a robust gradient formula. In particular, for given discontinuous states at the face, some kind of damping mechanism needs to be incorporated in the gradient formula. In the above formula, the second term, which is a quanitity of $O(h^2)$, acts as damping. In fact, the resulting viscous flux can be easily split into two parts (if one wishes) to identify the corresponding consistent and damping terms. In the next section, we propose a method in which the damping term is directly introduced into the numerical flux for the viscous term.

3.1.4. Viscous Flux via Upwind Flux

We can directly extend the hyperbolic approach: discretize an equivalent hyperbolic system for the viscous term and extract a viscous scheme from the result. Consider the following hyperbolic model for the viscous term proposed in Ref. [18]:

$$\frac{\partial \rho}{\partial t} = 0, \frac{\partial (\rho u)}{\partial t} = \frac{\partial \tau}{\partial x}, \frac{\partial (\rho E)}{\partial t} = \frac{\partial(-\tau u + q)}{\partial x}, \quad \frac{\partial \tau}{\partial t} = \frac{\mu_v}{T_v}\left(\frac{\partial u}{\partial x} - \frac{\tau}{\mu_v}\right), \quad \frac{\partial q}{\partial t} = \frac{\mu_h}{T_h}\left(-\frac{1}{\gamma(\gamma-1)}\frac{\partial T}{\partial x} - \frac{q}{\mu_h}\right), \tag{3.15}$$

where μ_v and μ_h are the scaled viscosities,

$$\mu_v = \frac{4}{3}\mu, \quad \mu_h = \frac{\gamma\mu}{Pr}, \tag{3.16}$$

T_v and T_h are the relaxation times associated with the viscous stress and the heat flux. The relaxation time for the diffusion equation (2.8) is extended to define them as

$$T_v = \frac{h^2}{\alpha^2 \nu_v}, \quad T_h = \frac{h^2}{\alpha^2 \nu_h}, \tag{3.17}$$

where ν_v and ν_h are the kinematic viscosities: $\nu_v = \mu_v/\rho$ and $\nu_h = \mu_h/\rho$. Note that the relaxation times are of $O(h^2)$ here, whereas they are of $O(1)$ in Ref. [18]. Note also that the system is not equivalent to the Navier-Stokes equations because τ and q are not precisely equal to the viscous stress and the heat flux, respectively, except in a steady state or in the limit: $T_v, T_h \to 0$. That is, their evolution equations are the possible source of inconsistency between the two models. To discretize the hyperbolic viscous system, we follow Ref. [18] and cast the system in the preconditioned conservative form:

$$\mathbf{P}^{-1}\frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{F}^v}{\partial x} = \mathbf{S}, \tag{3.18}$$

where

$$
\mathbf{V} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \\ \tau \\ q \end{bmatrix}, \quad \mathbf{F}^v = \begin{bmatrix} 0 \\ -\tau \\ -\tau u + q \\ -u \\ \dfrac{a^2}{\gamma(\gamma-1)} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau/\mu_v \\ q/\mu_h \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & T_v/\mu_v & 0 \\ 0 & 0 & 0 & 0 & T_h/\mu_h \end{bmatrix}. \tag{3.19}
$$

As shown in Ref. [18], this is a hyperbolic system having the the following eigenvalues:

$$
\lambda_1 = -a_v, \quad \lambda_2 = a_v, \quad \lambda_3 = -a_h, \quad \lambda_4 = a_h, \quad \lambda_5 = 0, \tag{3.20}
$$

where a_v and a_h are the viscous and heating speeds defined by

$$
a_v = \sqrt{\frac{\nu_v}{T_v}}, \quad a_h = \sqrt{\frac{\nu_h}{T_h}}. \tag{3.21}
$$

The speed a_v is associated with the viscous stress; it is called the viscous wave. On the other hand, a_h is associated with the heat flux; it is called the heating wave. It can be shown that the associated right-eigenvectors are linearly independent [18]. The system is, therefore, a hyperbolic system describing isotropic viscous and heating waves. Any numerical flux suitable for hyperbolic systems can be employed for the hyperbolic viscous system (i.e., we have many options here, much more than the choices for the gradient formula!). Here, we employ the viscous part of the upwind flux proposed in Ref. [18]:

$$
\mathbf{F}^v_{j+1/2} = \frac{1}{2}\left[\mathbf{F}^v(\mathbf{V}_L) + \mathbf{F}^v(\mathbf{V}_R)\right] - \frac{1}{2}\mathbf{P}^{-1}\left|\mathbf{P}\mathbf{A}^v\right|\Delta\mathbf{V}, \tag{3.22}
$$

where $\mathbf{A}^v = \partial\mathbf{F}^v/\partial\mathbf{V}$, and $\Delta\mathbf{V} = \mathbf{V}_R - \mathbf{V}_L$. The coefficient matrix of the dissipation term is evaluated by the arithmetic averages. Note that this flux is an upwind flux: a viscous part of the upwind Navier-Stokes flux proposed in Ref. [18]. The numerical flux for the hyperbolic system has been completely defined. It is time to derive a viscous flux. We derive it by ignoring the 4th and 5th components from the upwind flux. The result is

$$
\mathbf{F}^{vis}_{j+1/2} = \frac{1}{2}\left[\mathbf{F}^{vis}(\mathbf{U}_L, \nabla\mathbf{U}_L) + \mathbf{F}^{vis}(\mathbf{U}_R, \nabla\mathbf{U}_R)\right] - \frac{1}{2}\Delta\mathbf{F}^{vis}, \tag{3.23}
$$

where $\Delta\mathbf{F}^{vis}$ denotes the damping term given by

$$
\Delta\mathbf{F}^{vis} = \begin{bmatrix} 0 \\ \rho a_v \Delta u \\ \rho a_v u \Delta u + \dfrac{\rho a_h \Delta T}{\gamma(\gamma-1)} + \dfrac{\tau\Delta\tau}{\rho a_h(Pr_n+1)} \end{bmatrix}. \tag{3.24}
$$

We evaluate τ and q at nodes by Equation (3.3) with the gradients of the numerical solution, leading, for second-order reconstruction schemes, to the reconstructed gradients, $\nabla\mathbf{U}_{L/R} = \nabla\mathbf{U}_{j/k}$ (this is necessary because we do not store them as unknowns). Note that ΔT is the temperature jump defined by

$$
\Delta T \equiv \frac{1}{\rho}\left[\gamma\Delta p - a^2\Delta\rho\right], \tag{3.25}
$$

and Pr_n is the ratio of the viscous and heating wave speeds,

$$
Pr_n \equiv \frac{a_v}{a_h}. \tag{3.26}
$$

The resulting scheme will be a consistent time-accurate Navier-Stokes scheme because the viscous stress and the heat flux in the consistent term are evaluated consistently by the reconstructed gradients. Note that the consistent term contains merely the arithmetic average for the viscous stress and the heat flux. It is well known that the arithmetic average flux often fails to damp high-frequency errors and some mechanism needs to be incorporated for the high-frequency damping [19,20]. A useful feature of this hyperbolic system approach is that the term playing such a role, i.e., the damping term $\Delta\mathbf{F}^{vis}$, is automatically incorporated via the dissipation term of the upwind flux, which otherwise needs to be introduced by a special (typically method-dependent) technique.

Observe that the resulting viscous flux has a very similar structure to the inviscid flux. We can combine the viscous and inviscid fluxes into the full Navier-Stokes flux:

$$\mathbf{F}_{j+1/2} = \mathbf{F}^{inv}_{j+1/2} + \mathbf{F}^{vis}_{j+1/2} = \frac{1}{2}\left[\mathbf{F}(\mathbf{U}_L, \nabla\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R, \nabla\mathbf{U}_R)\right] - \frac{1}{2}\Delta\mathbf{F}, \tag{3.27}$$

where $\Delta\mathbf{F} = \Delta\mathbf{F}^{inv} + \Delta\mathbf{F}^{vis}$. The finite-volume Navier-Stokes scheme can be, therefore, systematically coded as a single loop over faces, computing the full Navier-Stokes flux at each face for given left and right values: $(\mathbf{U}_L, \nabla\mathbf{U}_L)$ and $(\mathbf{U}_R, \nabla\mathbf{U}_R)$. The time step restriction on the forward Euler explicit scheme is given by

$$\Delta t = \text{CFL}\frac{h}{\max(|u| + a + a_h)_j}, \tag{3.28}$$

where CFL is the Courant-Friedrichs-Lewy number less than or equal to 1, and the denominator is the maximum wave speed expressed as the sum of the maximum inviscid wave speed and the maximum viscous wave speed ($a_v < a_h$ for air). This is nothing but the CFL condition for a numerical scheme solving the hyperbolic Navier-Stokes system [18]. However, it should be noted that $a_h = O(1/h)$ in this work whereas $a_h = O(1)$ in Ref. [18]. Therefore, the time step is $O(h^2)$ for the Navier-Stokes scheme considered here. In effect, the stability condition is derived also from the discretization of the hyperbolic system.

3.2. Two Dimensions

3.2.1. Navier-Stokes Equations in Two Dimensions

Consider the compressible Navier-Stokes equations in two dimensions:

$$\frac{\partial\mathbf{U}}{\partial t} + \frac{\partial\mathbf{F}}{\partial x} + \frac{\partial\mathbf{G}}{\partial y} = 0, \tag{3.29}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}, \quad \mathbf{F} = \mathbf{F}^{inv} + \mathbf{F}^{vis} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{bmatrix} + \begin{bmatrix} 0 \\ -\tau_{xx} \\ -\tau_{xy} \\ -\tau_{xx}u - \tau_{xy}v + q_x \end{bmatrix}, \tag{3.30}$$

$$\mathbf{G} = \mathbf{G}^{inv} + \mathbf{G}^{vis} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{bmatrix} + \begin{bmatrix} 0 \\ -\tau_{yx} \\ -\tau_{yy} \\ -\tau_{yx}u - \tau_{yy}v + q_y \end{bmatrix}, \tag{3.31}$$

Here, v denotes the y-component of the velocity, τ_{xx}, τ_{xy}, and τ_{yy} denote the viscous stresses, and q_x and q_y denote the heat fluxes:

$$\tau_{xx} = \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right), \quad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{yy} = \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right), \tag{3.32}$$

$$q_x = -\frac{\mu}{Pr(\gamma-1)}\frac{\partial T}{\partial x}, \quad q_y = -\frac{\mu}{Pr(\gamma-1)}\frac{\partial T}{\partial y}. \tag{3.33}$$

All the quantities are understood as nondimensionalized as in Section 3.1.1. The viscosity is given by the scaled form of the Sutherland law (3.4).

3.2.2. Finite-Volume Discretization

We discretize the Navier-Stokes system by the node-centered edge-based finite-volume method:

$$\frac{d\mathbf{U}_j}{dt} = -\frac{1}{V_j} \sum_{k \in \{K_j\}} \mathbf{\Phi}_{jk} A_{jk}, \tag{3.34}$$

where $\mathbf{\Phi}_{jk}$ is a numerical flux defined at the midpoint of the edge along the directed area vector (see Figure 2.2). For second-order accuracy, we reconstruct the solution gradients at nodes in the primitive variables, (ρ, u, v, p), and extrapolate the solution to the edge-midpoint. On the boundary, a suitable boundary flux is applied with the linearity-preserving quadrature formulas [9, 23] (see Appendices of Ref. [9] for a comprehensive list of linearity-preserving quadrature formulas in both two and three dimensions). The numerical flux can be written as a sum of the inviscid and viscous parts:

$$\mathbf{\Phi}_{jk} = \mathbf{\Phi}_{jk}^{inv} + \mathbf{\Phi}_{jk}^{vis}. \tag{3.35}$$

We employ the Roe flux for the inviscid part:

$$\mathbf{\Phi}_{jk}^{inv} = \frac{1}{2}\left[\mathbf{H}_{jk}^{inv}\left(\mathbf{U}_L\right) + \mathbf{H}_{jk}^{inv}\left(\mathbf{U}_R\right)\right] - \frac{1}{2}\Delta\mathbf{H}^{inv}, \tag{3.36}$$

where $\Delta\mathbf{H}^{inv} = |\hat{\mathbf{A}}_n^{inv}|(\mathbf{U}_R - \mathbf{U}_L)$, and $\mathbf{U}_L$ and $\mathbf{U}_R$ are the extrapolated solution vectors to the edge-midpoint from the nodes, j and k, respectively. The absolute Jacobian, $|\hat{\mathbf{A}}_n^{inv}|$, is defined based on the directed area vector and the Roe-averages, and $\mathbf{H}_{jk}^{inv}$ is the physical inviscid flux projected along the directed area vector, $\mathbf{H}_{jk}^{inv} = \left[\mathbf{F}^{inv}, \mathbf{G}^{inv}\right] \cdot \hat{\mathbf{n}}_{jk}$. The viscous flux, $\mathbf{\Phi}_{jk}^{vis}$, remains to be defined. We construct the viscous flux by extending the diffusion scheme in Section 2.2. Again, there are two ways to do it.

3.2.3. Viscous Flux via Gradient Formula

We consider directly evaluating the physical viscous flux projected along the face direction, $\hat{\mathbf{n}}_{jk} = (n_x, n_y)$, at the interface:

$$\mathbf{\Phi}_{jk}^{vis} = \mathbf{H}_{jk}^{vis}(\mathbf{U}_{jk}, \nabla\mathbf{U}_{jk}) = \left[\begin{array}{c} 0 \\ \\ -\tau_{nx} \\ \\ -\tau_{ny} \\ \\ -(\tau_{nx}u + \tau_{ny}v) + q_n \end{array}\right]_{jk}, \tag{3.37}$$

where

$$\mathbf{H}_{jk}^{vis} = \left[\mathbf{F}^{vis}, \mathbf{G}^{vis}\right] \cdot \hat{\mathbf{n}}_{jk}, \quad \tau_{nx} = \tau_{xx}n_x + \tau_{xy}n_y, \quad \tau_{ny} = \tau_{xy}n_x + \tau_{yy}n_y, \quad q_n = q_y n_x + q_x n_y. \tag{3.38}$$

All quantities in the above expressions need to be computed at the interface. The velocity components and the viscosity may be evaluated by the arithmetic averages:

$$u_{jk} = \frac{u_L + u_R}{2}, \quad v_{jk} = \frac{v_L + v_R}{2}, \quad \mu_{jk} = \mu\left(\frac{T_L + T_R}{2}\right). \tag{3.39}$$

To evaluate the velocity and temperature gradients required for $(\tau_{xx}, \tau_{xy}, \tau_{yy}, q_x, q_y)_{jk}$, we apply the gradient formula (2.21):

$$\nabla u_{jk} = \overline{\nabla u}_{jk} + \frac{\alpha}{2L_r}\left(u_R - u_L\right)\hat{\mathbf{n}}_{jk}, \quad \nabla v_{jk} = \overline{\nabla v}_{jk} + \frac{\alpha}{2L_r}\left(v_R - v_L\right)\hat{\mathbf{n}}_{jk}, \quad \nabla T_{jk} = \frac{\gamma\nabla p_{jk} - a_{jk}^2\nabla\rho_{jk}}{\rho_{jk}}, \tag{3.40}$$

where ρ_{jk} and a_{jk} are the density and the speed of sound at the interface computed by the arithmetic averages of the density and the pressure, and

$$\nabla\rho_{jk} = \overline{\nabla\rho}_{jk} + \frac{\alpha}{2L_r}\left(\rho_R - \rho_L\right)\hat{\mathbf{n}}_{jk}, \quad \nabla p_{jk} = \overline{\nabla p}_{jk} + \frac{\alpha}{2L_r}\left(p_R - p_L\right)\hat{\mathbf{n}}_{jk}. \tag{3.41}$$

The viscous stresses and the heat fluxes are then computed by Equations (3.32) and (3.33). This completes the construction of the numerical viscous flux. This approach is a widely-used method for extending a diffusion scheme to a viscous scheme; but the application of the gradient formula (2.21) is new. Again, this approach requires a gradient formula carefully designed to ensure high-frequency damping. In the above formula, the second term, which is a quanitity of $O(h^m)$ for m-th order accurate reconstructed nodal gradients, acts as damping. As in one dimension, the resulting viscous flux can be easily split into two parts (if one wishes) to identify the corresponding consistent and damping terms. On the other hand, in the hyperbolic approach, the damping term is directly introduced into the numerical flux for the viscous term.

3.2.4. Viscous Flux via Upwind Flux

We begin by defining a hyperbolic viscous system. Following Ref. [18], we introduce the evolution equations for the viscous stresses and the heat fluxes,

$$\frac{\partial \tau_{xx}}{\partial t} = \frac{\mu_v}{T_v}\left(\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} - \frac{\tau_{xx}}{\mu_v}\right), \quad \frac{\partial \tau_{xy}}{\partial t} = \frac{\mu_v}{T_v}\left(\frac{3}{4}\frac{\partial u}{\partial y} + \frac{3}{4}\frac{\partial v}{\partial x} - \frac{\tau_{xy}}{\mu_v}\right), \quad \frac{\partial \tau_{yy}}{\partial t} = \frac{\mu_v}{T_v}\left(\frac{\partial v}{\partial y} - \frac{1}{2}\frac{\partial u}{\partial x} - \frac{\tau_{yy}}{\mu_v}\right), \tag{3.42}$$

$$\frac{\partial q_x}{\partial t} = \frac{\mu_h}{T_h}\left(-\frac{1}{\gamma(\gamma-1)}\frac{\partial T}{\partial x} - \frac{q_x}{\mu_h}\right), \quad \frac{\partial q_y}{\partial t} = \frac{\mu_h}{T_h}\left(-\frac{1}{\gamma(\gamma-1)}\frac{\partial T}{\partial y} - \frac{q_y}{\mu_h}\right), \tag{3.43}$$

and construct the following first-order viscous system:

$$\mathbf{P}^{-1}\frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{F}^v}{\partial x} + \frac{\partial \mathbf{G}^v}{\partial y} = \mathbf{S}, \tag{3.44}$$

where

$$\mathbf{V} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{yy} \\ q_x \\ q_y \end{bmatrix}, \quad \mathbf{F}^v = \begin{bmatrix} 0 \\ -\tau_{xx} \\ -\tau_{xy} \\ -\tau_{xx}u - \tau_{xy}v + q_x \\ -u \\ -3v/4 \\ u/2 \\ \dfrac{a^2}{\gamma(\gamma-1)} \\ 0 \end{bmatrix}, \quad \mathbf{G}^v = \begin{bmatrix} 0 \\ -\tau_{xy} \\ -\tau_{yy} \\ -\tau_{xy}u - \tau_{yy}v + q_y \\ v/2 \\ -3u/4 \\ -v \\ 0 \\ \dfrac{a^2}{\gamma(\gamma-1)} \end{bmatrix}, \tag{3.45}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tau_{xx}/\mu_v \\ \tau_{xy}/\mu_v \\ \tau_{yy}/\mu_v \\ q_x/\mu_h \\ q_y/\mu_h \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_v/\mu_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_v/\mu_v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_v/\mu_v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_h/\mu_h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_h/\mu_h \end{bmatrix}. \tag{3.46}$$

The relaxation times, T_v and T_h, are defined by

$$T_v = \frac{L_r^2}{\alpha^2 \nu_v}, \quad T_h = \frac{L_r^2}{\alpha^2 \nu_h}. \tag{3.47}$$

The length scale, L_r, is defined as in Equation (2.19). Note that the relaxation times are of $O(h^2)$, not of $O(1)$. As shown in Ref. [18], this first-order viscous system is a hyperbolic system having the following eigenvalues:

$$\lambda_1^v = -a_{nv}, \quad \lambda_2^v = a_{nv}, \quad \lambda_3^v = -a_{mv}, \quad \lambda_4^v = a_{mv}, \quad \lambda_5^v = -a_h, \quad \lambda_6^v = a_h, \quad \lambda_{7,8,9}^v = 0, \tag{3.48}$$

where

$$a_{nv} = \sqrt{\frac{\nu_v}{T_v}}, \quad a_{mv} = \sqrt{\frac{3\nu_v}{4T_v}}, \quad a_h = \sqrt{\frac{\nu_h}{T_h}}. \tag{3.49}$$

The speed a_{nv} is associated with the normal viscous stress; it is called the normal viscous wave. On the other hand, a_{mv} is associated with the shear viscous stress; it is called the shear viscous wave. As in one dimension, a_h is the speed for the heating wave. The corresponding right-eigenvectors can be shown to be linearly independent. The system is, therefore, a hyperbolic system describing isotropic normal/shear viscous and heating waves. Various choices are possible for constructing a numerical flux for the hyperbolic system (i.e., again, we have many options here, much more than the choices for the gradient formula!). In this paper, we employ the viscous part of the upwind flux proposed in Ref. [18]:

$$\mathbf{\Phi}_{jk}^v \;=\; \frac{1}{2}\left[\mathbf{H}_{jk}^v(\mathbf{V}_L) + \mathbf{H}_{jk}^v(\mathbf{V}_R)\right] - \frac{1}{2}\mathbf{P}^{-1}\left|\mathbf{P}\mathbf{A}^v\right|\Delta\mathbf{V}, \tag{3.50}$$

where $\mathbf{H}_{jk}^v = [\mathbf{F}^v, \mathbf{G}^v]\cdot\hat{\mathbf{n}}_{jk}$, and $\mathbf{A}^v = \partial \mathbf{H}_{jk}^v/\partial \mathbf{V}$. The coefficient of the dissipation term is evaluated by the arithmetic averages. We emphasize that this flux is an upwind flux: a viscous part of the upwind Navier-Stokes flux proposed in Ref. [18]. The numerical flux for the hyperbolic system has been completely defined. It is time to derive a viscous flux. By ignoring the 5th, 6th, 7th, 8th, and 9th components from the upwind flux, we obtain

$$\mathbf{\Phi}_{jk}^{vis} \;=\; \frac{1}{2}\left[\mathbf{H}_{jk}^{vis}(\mathbf{U}_L, \nabla\mathbf{U}_L) + \mathbf{H}_{jk}^{vis}(\mathbf{U}_R, \nabla\mathbf{U}_R)\right] - \frac{1}{2}\Delta\mathbf{H}^{vis}, \tag{3.51}$$

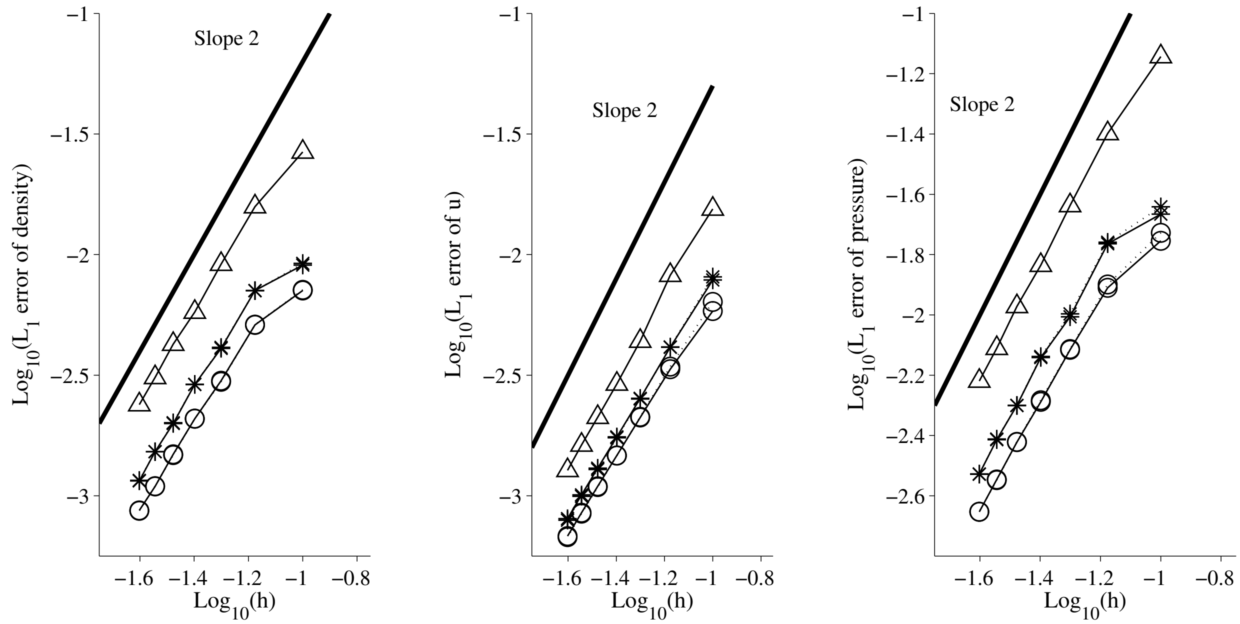
where the damping term, $\Delta\mathbf{H}^{vis}$, can be shown to be a straightforward extension of the one-dimensional damping term. Note that the viscous stresses and heat fluxes are evaluated at nodes by Equations (3.32) and (3.33) with the gradients of the numerical solution. For second-order reconstruction schemes, these gradients are equivalent to the reconstructed gradients: $\nabla\mathbf{U}_L = \nabla\mathbf{U}_j$ and $\nabla\mathbf{U}_R = \nabla\mathbf{U}_k$. The resulting scheme will be a consistent time-accurate Navier-Stokes scheme because the viscous stresses and the heat fluxes in the consistent term of the numerical flux are computed consistently by the reconstructed gradients. Note that the consistent term is merely the arithmetic average for the viscous stresses and the heat fluxes; the viscous damping term, $\Delta\mathbf{H}^{vis}$, has been automatically incorporated via the dissipation term of the upwind flux. As in one dimension, we can combine the derived viscous flux and the Roe flux to form a full Navier-Stokes flux:

$$\mathbf{\Phi}_{jk} = \mathbf{\Phi}_{j+1/2}^{inv} + \mathbf{\Phi}_{j+1/2}^{vis} = \frac{1}{2}\left[\mathbf{H}_{jk}(\mathbf{U}_L, \nabla\mathbf{U}_L) + \mathbf{H}_{jk}(\mathbf{U}_R, \nabla\mathbf{U}_R)\right] - \frac{1}{2}\Delta\mathbf{H}, \tag{3.52}$$

where $\mathbf{H}_{jk} = [\mathbf{F}, \mathbf{G}]\cdot\hat{\mathbf{n}}_{jk}$ and $\Delta\mathbf{H} = \Delta\mathbf{H}^{inv} + \Delta\mathbf{H}^{vis}$. The edge-based finite-volume scheme can be, therefore, systematically coded as a single loop over faces, computing the full Navier-Stokes flux at each face for given left and right values: $(\mathbf{U}_L, \nabla\mathbf{U}_L)$ and $(\mathbf{U}_R, \nabla\mathbf{U}_R)$. Also, the stability condition on the forward Euler explicit scheme is derived from the upwind scheme applied to the hyperbolic Navier-Stokes system [18]. It is defined as the minimum of the local time-step, Δt_j, restricted by the local CFL condition:

$$\Delta t_j = \text{CFL}\frac{2V_j}{\displaystyle\sum_{k\in\{K_j\}}(|u_n| + a + a_h)_j A_{jk}}, \quad \text{CFL} \le 1. \tag{3.53}$$

It is important to note that $a_h = O(1/h)$ here whereas $a_h = O(1)$ in Ref. [18], and therefore, the time step is $O(h^2)$ for the Navier-Stokes scheme considered here.



(a) L_1 error of ρ. (b) L_1 error of u. (c) L_1 error of p.

Figure 3.1. Error convergence results for the main variables in the one-dimensional problem. Solid line: the gradient-based viscous flux with $\alpha = 0$ ($\triangle$), $\alpha = 2$ ($\star$), $\alpha = 8/3$ ($\circ$). Dashed line: the derived viscous flux with $\alpha = 2$ ($\star$), $\alpha = 8/3$ ($\circ$).

3.3. Remarks

In both one and two dimensions, the difference between the two viscous fluxes derived from the two approaches lies mainly in the linearization and the additional viscous-heating coupling term in the energy damping. The effect of the viscous-heat coupling in the damping term remains to be investigated; no noticeable differences have been observed for a simple test problem considered in this study. It is noted that the viscous flux derived from the upwind flux has a very similar structure to the inviscid flux and thus it can be very naturally integrated with the inviscid flux into a full Navier-Stokes flux in the form of Equations (3.27) and (3.52). Actually, the full Navier-Stokes flux can be derived directly from *the upwind Navier-Stokes flux* constructed in Ref. [18] by ignoring extra components. Note also that it is possible to derive other viscous fluxes by applying other numerical fluxes to the hyperbolic viscous system. Moreover, the numerical viscous fluxes constructed here can be directly employed in other methods: cell-centered finite-volume, discontinuous Galerkin and spectral volume/difference methods. In cell-centered finite-volume methods, these viscous fluxes are different from widely-used fluxes in that these viscous fluxes are applied precisely at the quadrature points on the control-volume boundary [9]. Finally, we emphasize that the construction of the viscous flux through a hyperbolic system is not limited to the finite-volume method. The main idea being the use of the hyperbolic system, it can be employed in any discretization method: simply discretize the hyperbolic system and derive a viscous discretization from the result. A successful numerical scheme for hyperbolic systems typically has a dissipation term, and it will turn into a damping term for the derived viscous scheme.

4. Numerical Results

Preliminary results are available for a viscous shock-structure problem. The exact solution can be computed by solving a pair of ordinary differential equations for the velocity and the temperature; see Ref. [24] for the description and visit `http://www.cfdbooks.com/cfdcodes.html` to download the source code used to generate the exact solution in this study. Accuracy comparison is made for the finite-volume Navier-Stokes schemes arising from the two approaches: the gradient-based viscous flux and the derived viscous flux. In all computations, we take $M_\infty = 3.5$, $Pr = 3/4$, $\gamma = 1.4$, $Re_\infty = 25$, and $T_\infty = 400$ [k]. Time integration is performed by the forward Euler time-stepping scheme with CFL$= 0.99$ until the divided residual, which is equivalent to the change in the

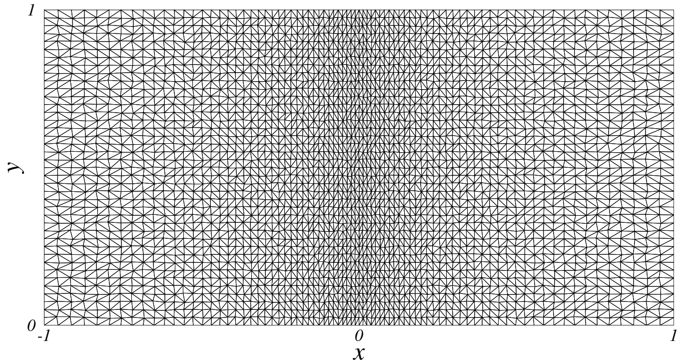


Figure 4.1. Irregular grid for the viscous shock-structure problem in two dimensions (3321 nodes).

solution divided by the time step, is reduced by six orders of magnitude in the L_1 norm.

4.1. One-Dimensional Problem

Using the exact solution as the initial solution, we integrate the Navier-Stokes equations toward the steady state. The domain is taken as $x = [-1, 1]$. All grids are uniformly spaced with 21, 31, 41, 51, 61, 71, 81 nodes. To fix the shock location, we keep the exact pressure at $x = 0.5$ for all grids. On the left/right boundary, the flux is computed by the Roe flux with the left/right state given by the exact solution.

Figure 4.1 shows the error convergence results. The results almost overlap, but solid lines are used for the Navier-Stokes scheme with the gradient-based viscous flux while dashed lines are used for the Navier-Stokes scheme with the derived viscous flux. It is observed that both Navier-Stokes schemes are second-order accurate. Also, it can be seen that the choice $\alpha = 8/3$, which makes the viscous scheme fourth-order accuracy, gives consistently more accurate results in both schemes as expected. Finally, the scheme with no damping (i.e., $\alpha = 0$) yields significantly less accurate solutions.

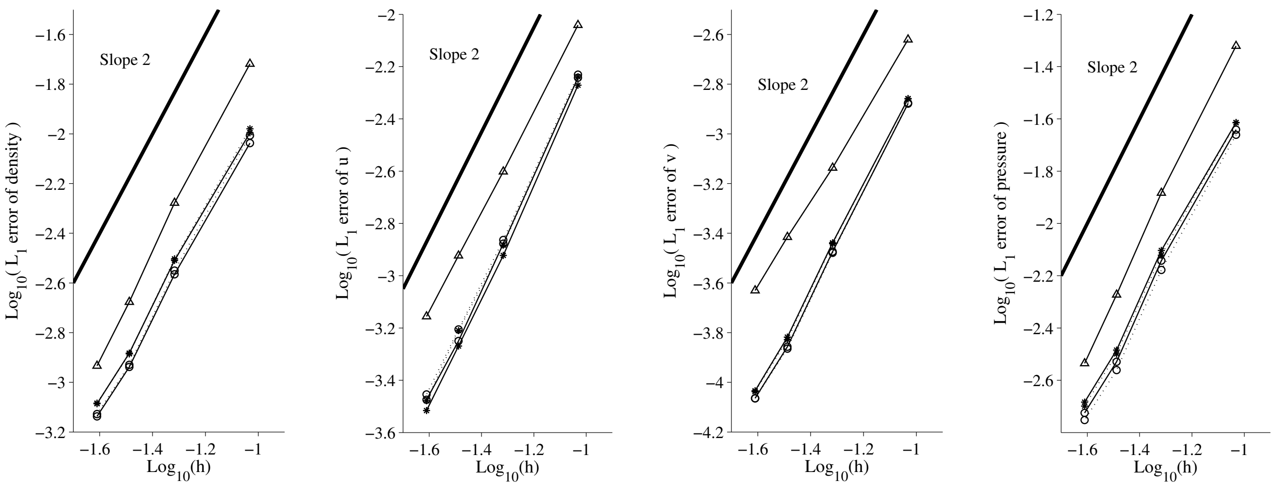
4.2. Two-Dimensional Problem

We consider the one-dimensional viscous shock-structure solution in a two-dimensional rectangular domain. Preliminary results are available for irregular triangular grids generated from 21x11, 41x21, 61x31, 81x41 structured grids by random diagonal splitting, random nodal perturbation, and stretching. In each grid, the nodes are clustered over the viscous shock as shown in Figure 4.2. Again, the exact solution is used as the initial solution. Also, similar internal pressure condition and boundary conditions are applied as in one dimension. The gradients are computed at nodes by the unweighted least-squares reconstruction.

Figure 4.3 shows the error convergence results. These results show that both Navier-Stokes schemes are second-order accurate for $\alpha = 0$, $\alpha = 1$, and $\alpha = 4/3$. However, the solutions are significantly inaccurate for $\alpha = 0$ compared with others. This failure is considered as due to the lack of damping. We observe a slight accuracy improvement with $\alpha = 4/3$ over $\alpha = 1$, but not very significant for this test problem. The impact of the damping coefficient on the solution accuracy is expected to be more significant on highly-skewed (typical adapted viscous) grids as demonstrated in Refs. [9, 17] for diffusion schemes.

5. Concluding Remarks

We have extended the diffusion scheme derived in Refs. [9, 17] to the Navier-Stokes equations in two different ways. One is a popular way of directly evaluating the gradients in the viscous flux. In order to employ this approach, we cast the diffusion scheme in the interface gradient form and identify the corresponding gradient formula. The other way is a direct extension of the hyperbolic approach. We employed the hyperbolic viscous system proposed in Ref. [18], discretized it by an upwind flux, and derived a viscous flux from the result. We demonstrated for the hyperbolic approach that the damping term, which is essential to robust and accurate viscous computations,



(a) L_1 error of ρ. (b) L_1 error of u. (c) L_1 error of v. (d) L_1 error of p.

Figure 4.2. Error convergence results for the two-dimensional problem. Solid line: the gradient-based viscous flux with $\alpha = 0$ ($\triangle$), $\alpha = 1$ ($\star$), $\alpha = 4/3$ ($\circ$). Dashed line: the derived viscous flux with $\alpha = 1$ ($\star$), $\alpha = 4/3$ ($\circ$).

is directly and automatically introduced into the viscous discretization. In either case, the damping parameter, α, has been shown to have an impact on the solution accuracy: more accurate solution with $\alpha = 8/3$ in one dimension and (although only very slightly) $\alpha = 4/3$ in two dimensions; the lack of damping ($\alpha = 0$) leads to inaccurate solutions. In order to illustrate the importance of the damping term, however, it is necessary to perform numerical experiments on highly-skewed grids. A successful demonstration will increase our confidence in applying derived viscous schemes to demanding applications such as fully-adapted viscous grids and unstructured hypersonic viscous simulations. Also, it should be noted that the values of α used in this study are based on the one-dimensional analysis in Refs. [9, 17]; more suitable values may be discovered by two-dimensional analysis.

We remark that the classification by the consistent and damping terms is just one useful way to look at viscous discretizations and there can be, of course, others. As discussed in Refs. [9,17], if the solution is continuous or made continuous [25,7] across the interface, then the damping term vanishes; but still the resulting scheme can be robust and accurate. In fact, the construction of a continuous solution across the interface is a very natural principle for diffusion [25]; it is just not immediately clear how it can be extended to other discretization methods, particularly to the residual-distribution method which is based on a continuous solution but requires a damping term [9, 17].

In this paper, we only considered the finite-volume method. However, the viscous flux constructed in this paper can be directly employed in other methods: discontinuous Galerkin and spectral volume/difference methods. It is also possible to derive other viscous fluxes by applying other numerical fluxes to the discretization of the hyperbolic viscous system, such as HLLC [26] or flux-vector splitting fluxes [27, 28, 29, 30]. Yet, this hyperbolic approach is a general approach applicable to various other discretization methods including the residual-distribution method. In any case, a robust and accurate viscous discretization endowed with a damping term will be derived. A challenge is to derive a Navier-Stokes scheme in one shot: discretize a hyperbolic model for the whole Navier-Stokes equations *based on its full eigen-structure.* The resulting scheme is expected to automatically incorporate a proper balance between the inviscid and viscous terms as demonstrated for a model equation in Ref. [31]. Finally, we remark that the hyperbolic model for the viscous term is not unique. Various other models can be proposed, and thereby even more various viscous discretizations may be derived. The way has just been paved for generating a greater variety of viscous discretizations than what we have today.

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