The Twilight Zone™

CFD

"Forbidden Capability"

It is often said that most great inventions begin as a crazy idea. But a crazy idea should be handled carefully. If you get too excited and drawn into the craziness, you'd be trapped and never be able to come back to the real world.

Dr. Hiro Nishikawa, a CFD researcher, encounters an unusual and 'amazing' capability of CFD, but people don't take it seriously. He struggles to prove that it is real and useful in an attempt to escape from the Twilight Zone.
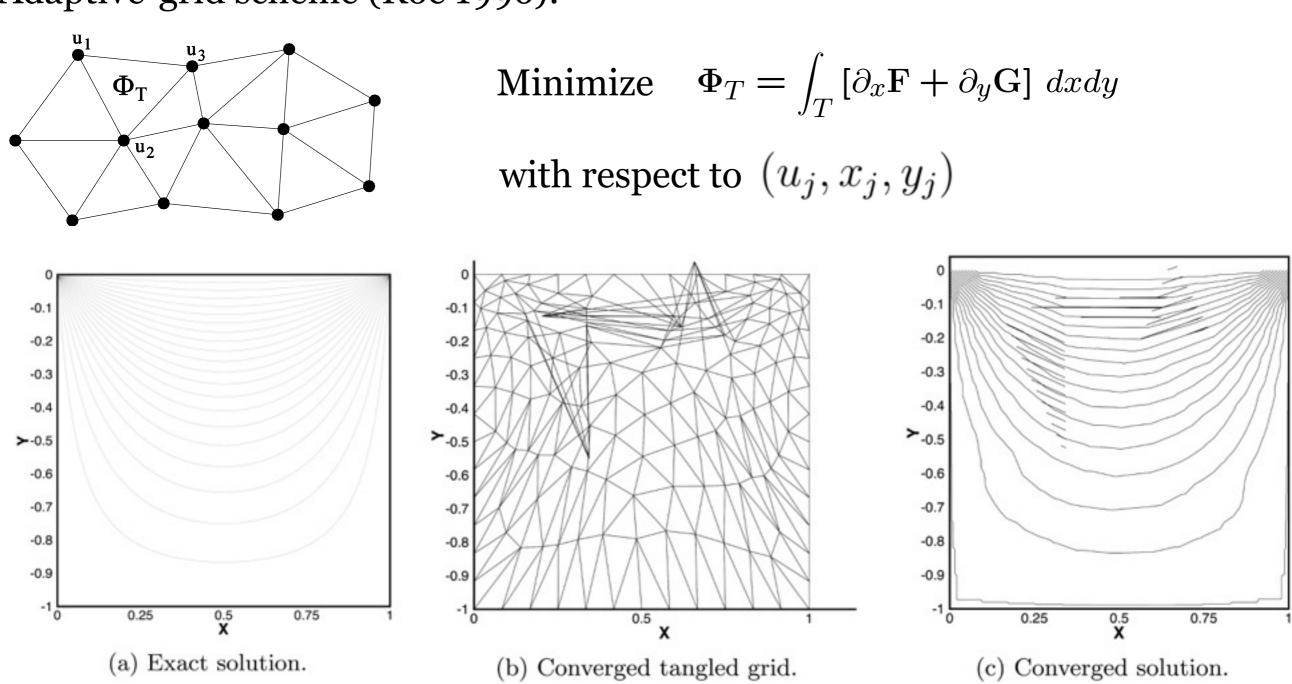
Adaptive-grid scheme (Roe 1996):

$$\text{Minimize} \quad \Phi_T = \int_T \left[\partial_x \mathbf{F} + \partial_y \mathbf{G} \right] \, dx dy$$

with respect to (u_j, x_j, y_j)

(a) Exact solution.

(b) Converged tangled grid.

(c) Converged solution.

Adaptive-grid scheme converged to a tangled mesh.
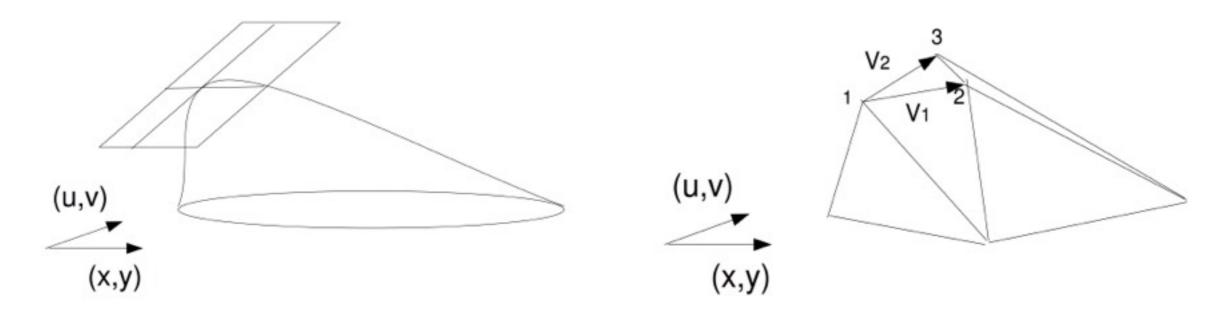
Solution of a pair of PDEs (2D) = Surface in 4D space.

The method adjusts the nodes to better approximate the surface.

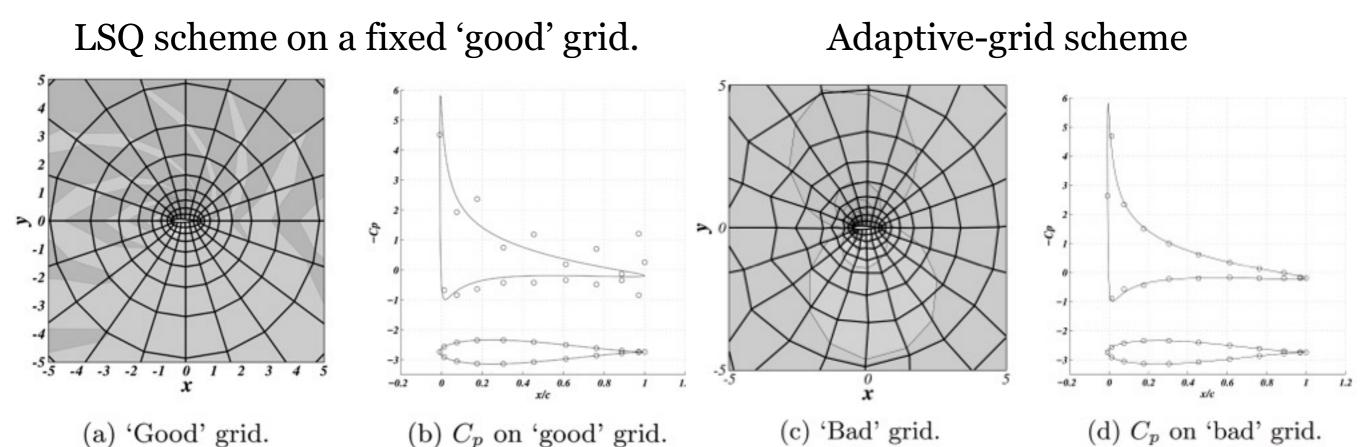Residual = Error in aligning a triangle tangent to the surface.

"A Geometric Interpretation of Fluctuations" 1997 (available at cfdnotes.com)

So, it doesn't matter whether a triangle is inverted or not.

"Grids and Solutions from Residual Minimization", PhD Thesis 2001
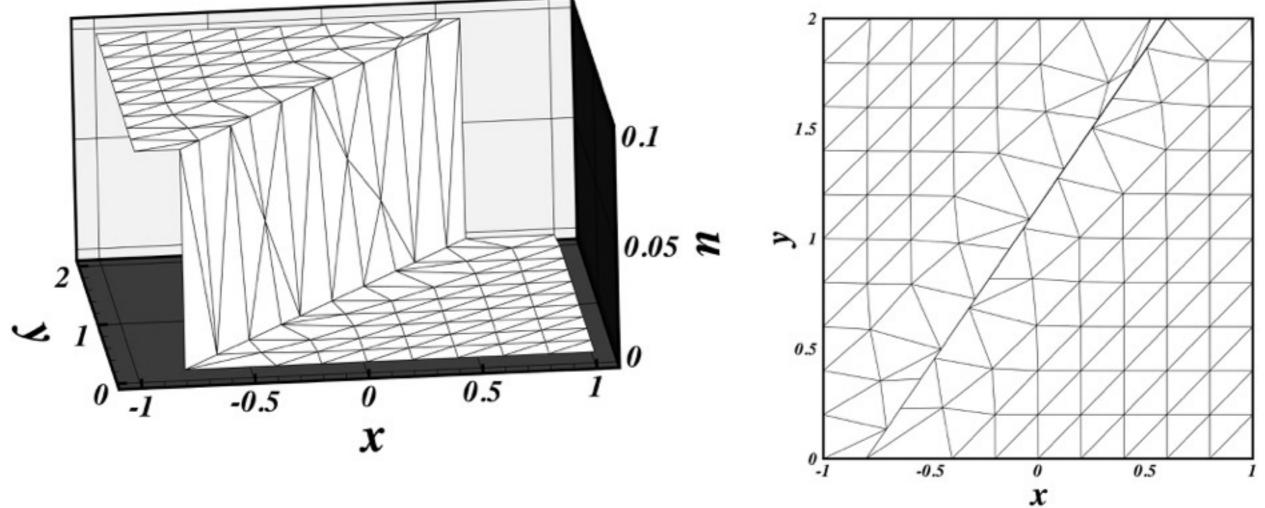
2001: Good grid or Bad grid

LSQ scheme on a fixed 'good' grid.   Adaptive-grid scheme

(a) 'Good' grid.   (b) C_p on 'good' grid.   (c) 'Bad' grid.   (d) C_p on 'bad' grid.

A bad grid can be a good grid for a bad scheme.

"Grids and Solutions from Residual Minimization", PhD Thesis 2001

Adaptive-grid scheme converged and captured
a shock perfectly with zero-volume triangles.

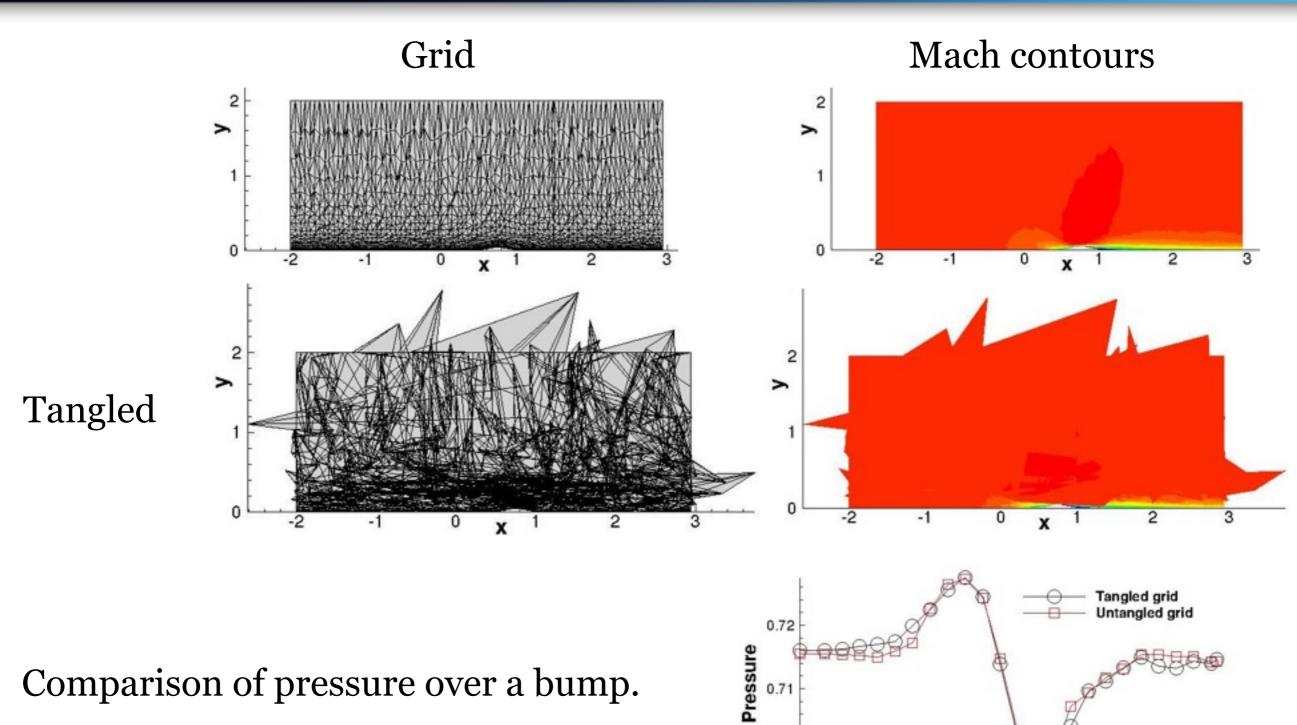"Grids and Solutions from Residual Minimization", PhD Thesis 2001

Grid

Mach contours

Tangled

Comparison of pressure over a bump.

CFD on a tangled grid is real

Truth is that he is not alone: Many other CFD researchers have a similar experience. However…

No formal publications are found.
He begins to understand why.
A big question is unanswered:

"So what?"

2016: Finally, asked a Good Question

How is it useful?

CFD on 'bad' grid indicates robustness, but it is almost always possible to 'fix' the grid.

Unless a scheme is proved to work with zero/neg-volume elements, and the use of them brings advantages, it generates no interests.

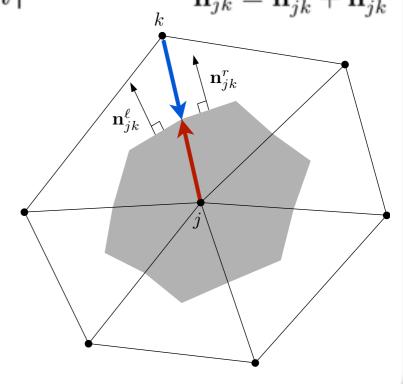Hiro began to make a move.

Edge-Based Discretization

NASA's FUN3D; Software Cradle's SC/Tetra; DLR Tau code, etc.

$$\mathrm{div}\, \mathbf{f} = 0 \quad \longrightarrow \quad \frac{1}{V_j} \sum_{k \in \{k_j\}} \phi_{jk} |\mathbf{n}_{jk}| = 0$$

$$\mathbf{n}_{jk} = \mathbf{n}_{jk}^{\ell} + \mathbf{n}_{jk}^{r}$$

Numerical flux at edge-midpoint

$$\phi_{jk} = \frac{1}{2}(\mathbf{f}_L + \mathbf{f}_R) \cdot \hat{\mathbf{n}}_{jk} - \frac{1}{2}|a_n|(u_R - u_L)$$

linearly extrapolated L and R states.

Edge-based formula is a special flux quadrature.

Facts about EB Scheme

2nd-order only on simplex(tria/tet)-element grids.

3rd-order with quadratic LSQ fits on simplex grids.
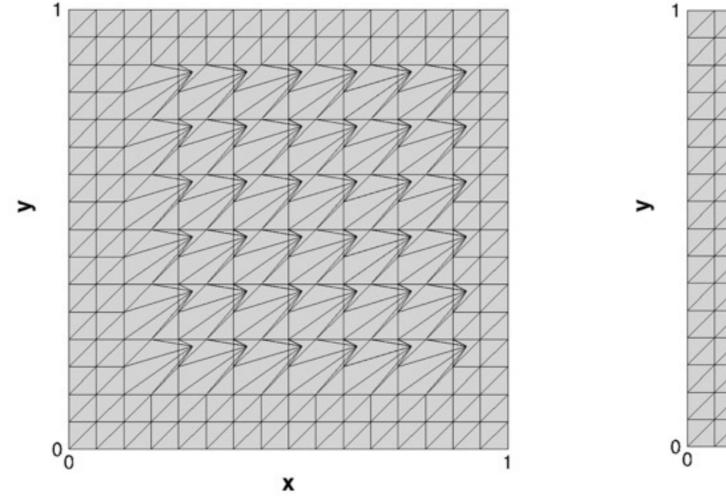
Theory : Nishikawa Liu, JCP2017

Two schemes:

1. **Conventional scheme: 2nd-order accurate.**

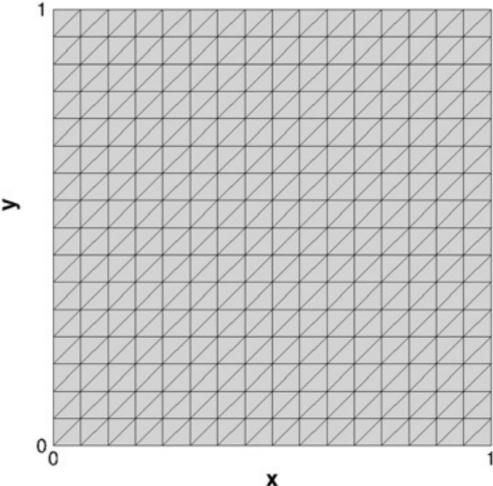2. **Hyperbolic scheme: 3rd-order for advection.**

We consider advection-diffusion and Burgers' equations only. Euler and NS schemes will be considered in future.

It can be mapped to a grid of all positive volumes.

Topologically equivalent.

All computed such that an element has a positive volume when its nodes are ordered counterclockwise.
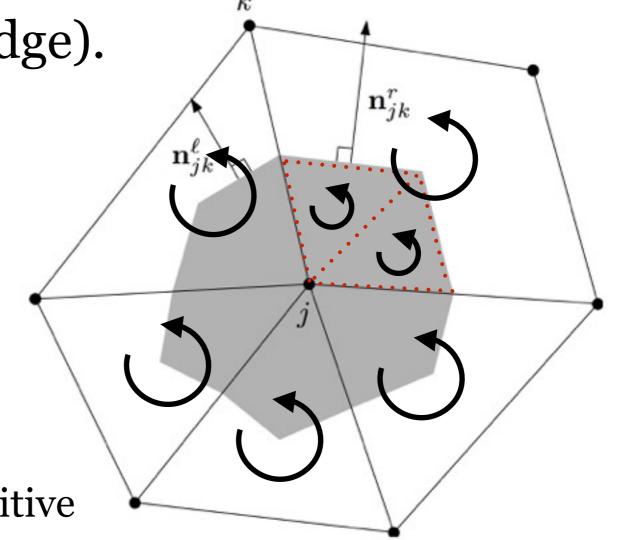
1. Directed area vector (per edge).

$$\mathbf{n}_{jk} = \mathbf{n}_{jk}^{\ell} + \mathbf{n}_{jk}^{r}$$

2. Dual volume (per node)

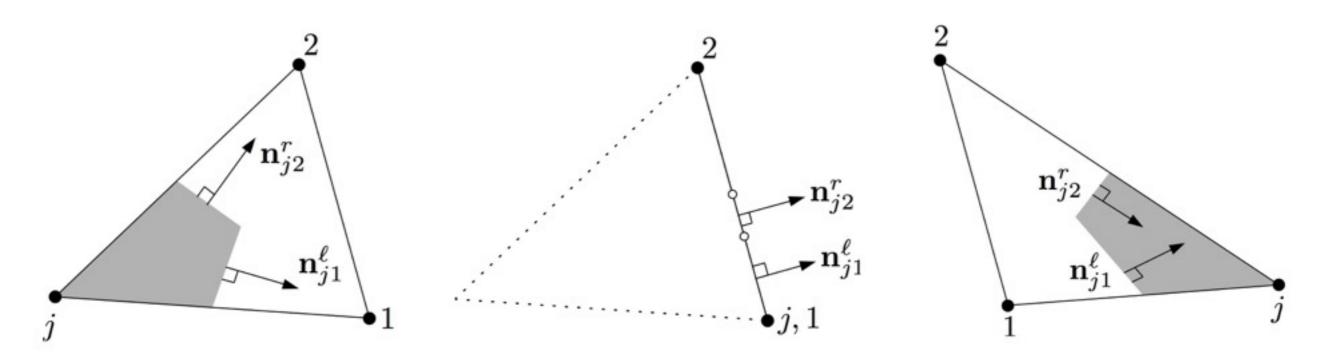Sum of the contributions from surrounding elements (triangles).

-> Signed volume: negative or positive

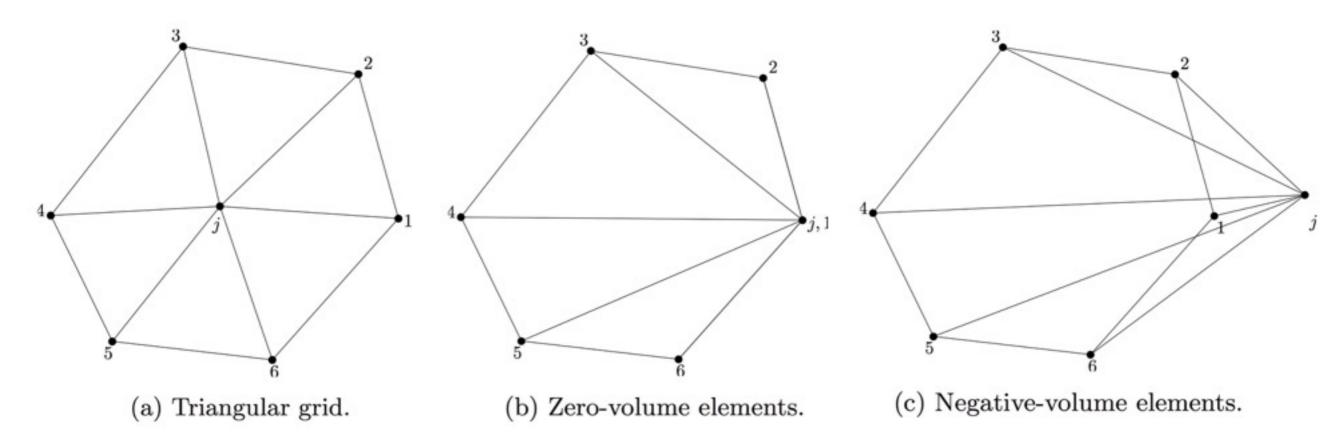**Do NOT modify existing codes.**

Face normal may be inverted, but OK, so that

$$\sum_{k \in \{k_j\}} \mathbf{n}_{jk} = 0,$$

Do not even check the local direction of njk.

Dual volume is independent of the location of j.

(a) Triangular grid.   (b) Zero-volume elements.   (c) Negative-volume elements.

Keep zero/negative-volume elements as they are, so that we maintain a consistency condition:

$$\sum V_j = \text{Total volume of the domain}$$

Negative Dual Volume

Linear advection: $\partial_x(au) + \partial_y(bu) = 0$

<u>Positive Dual volume:</u>

$$\sum_{k \in \{k_j\}} \phi_{jk}|\mathbf{n}_{jk}| = [\partial_x(au) + \partial_y(bu)]\,|V_j| + TE$$

<u>Negative Dual volume:</u>

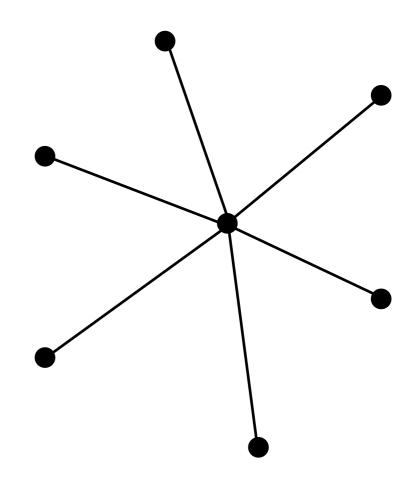$$\sum_{k \in \{k_j\}} \phi_{jk}|\mathbf{n}_{jk}| = -[\partial_x(au) + \partial_y(bu)]\,|V_j| + TE,$$

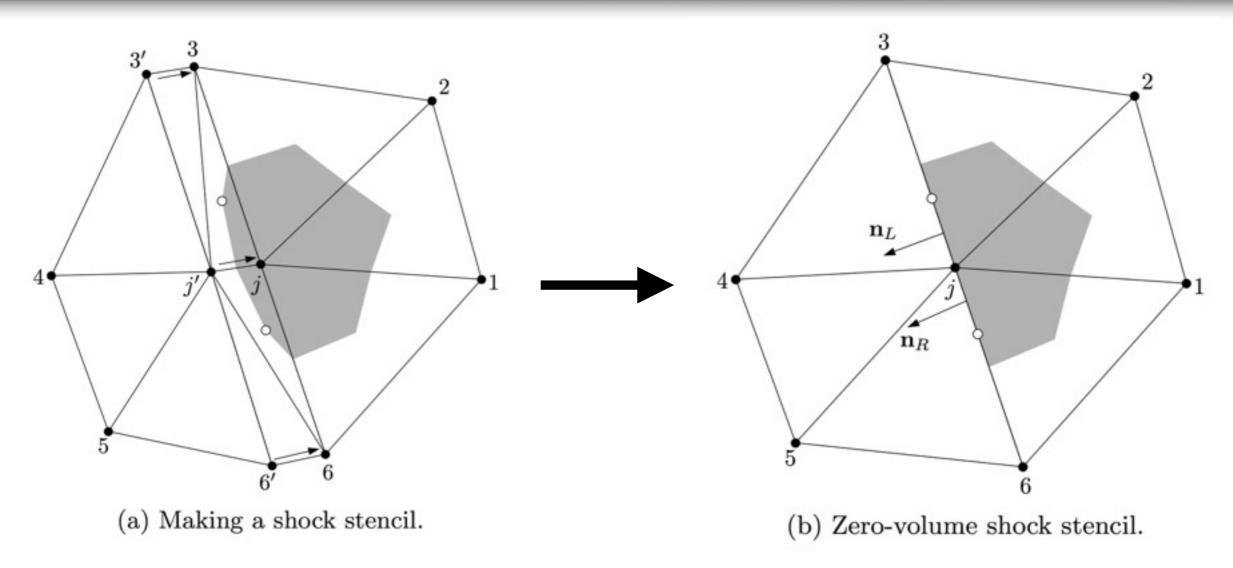**We'll be solving the target PDE with a negative sign.**

LSQ Gradients

LSQ gradients are independent of element orientation.

It works with zero/neg-volume.

Shock-Capturing

(a) Making a shock stencil.

(b) Zero-volume shock stencil.

EB residual reduces to the Rankine-Hugoniot:

$$Res_j = \left(|\bar{a}_n|\Delta u - \Delta f_n\right)|\mathbf{n}_L + \mathbf{n}_R|,$$

Implicit Solver

Jacobian-Free Newton's method

$$\mathbf{U}_h^{n+1} = \mathbf{U}_h^n + \Delta\mathbf{U}_h \qquad \frac{\partial\mathbf{Res}}{\partial\mathbf{U}_h}\Delta\mathbf{U}_h = -\mathbf{Res}(\mathbf{U}_h^n)$$
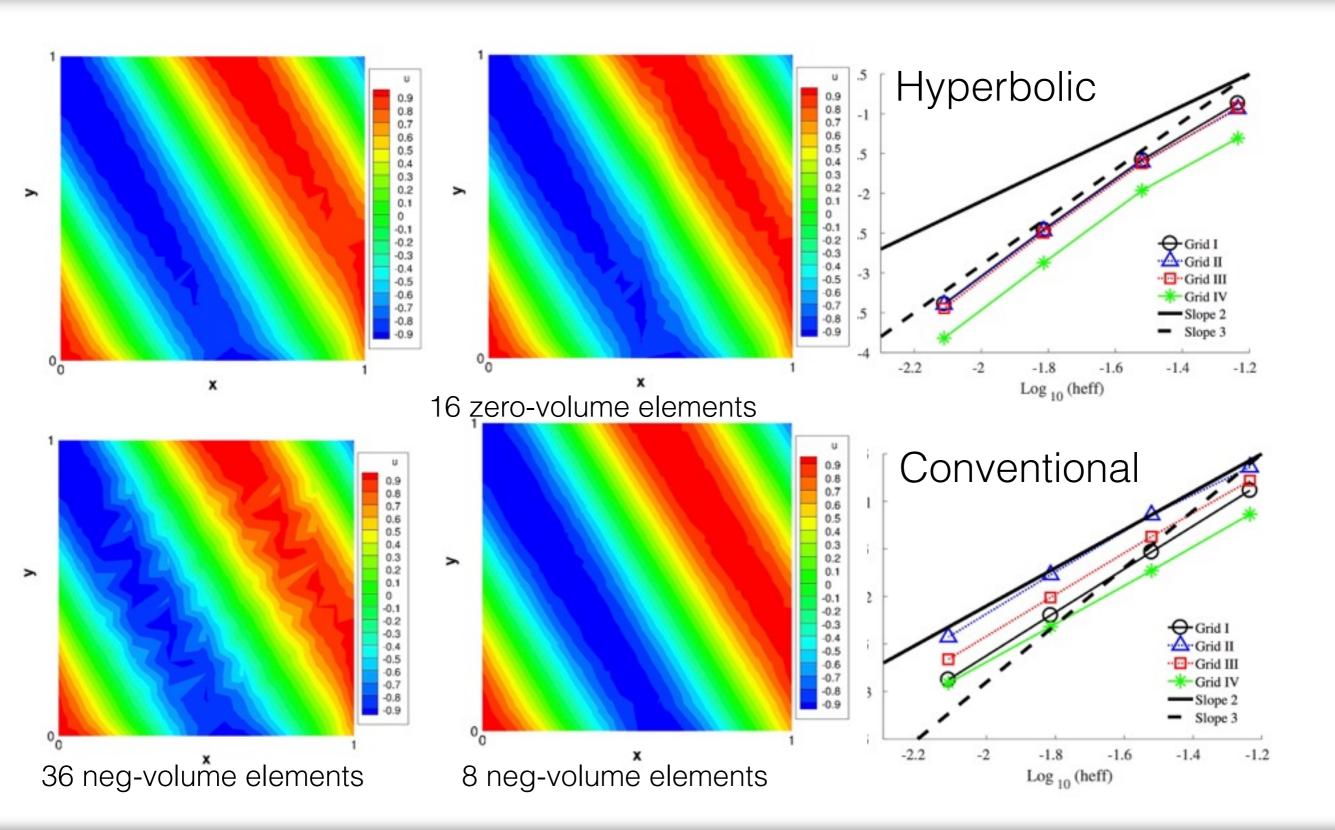
NOTE: There is no pseudo time term (CFL=infinity).

It converges in 10 iterations (max 17) for all test problems.

Accuracy Verification

Advection equation

16 zero-volume elements

Hyperbolic

Conventional

36 neg-volume elements

8 neg-volume elements

"Forbidden Capability"

Answers to "How can they be useful?"
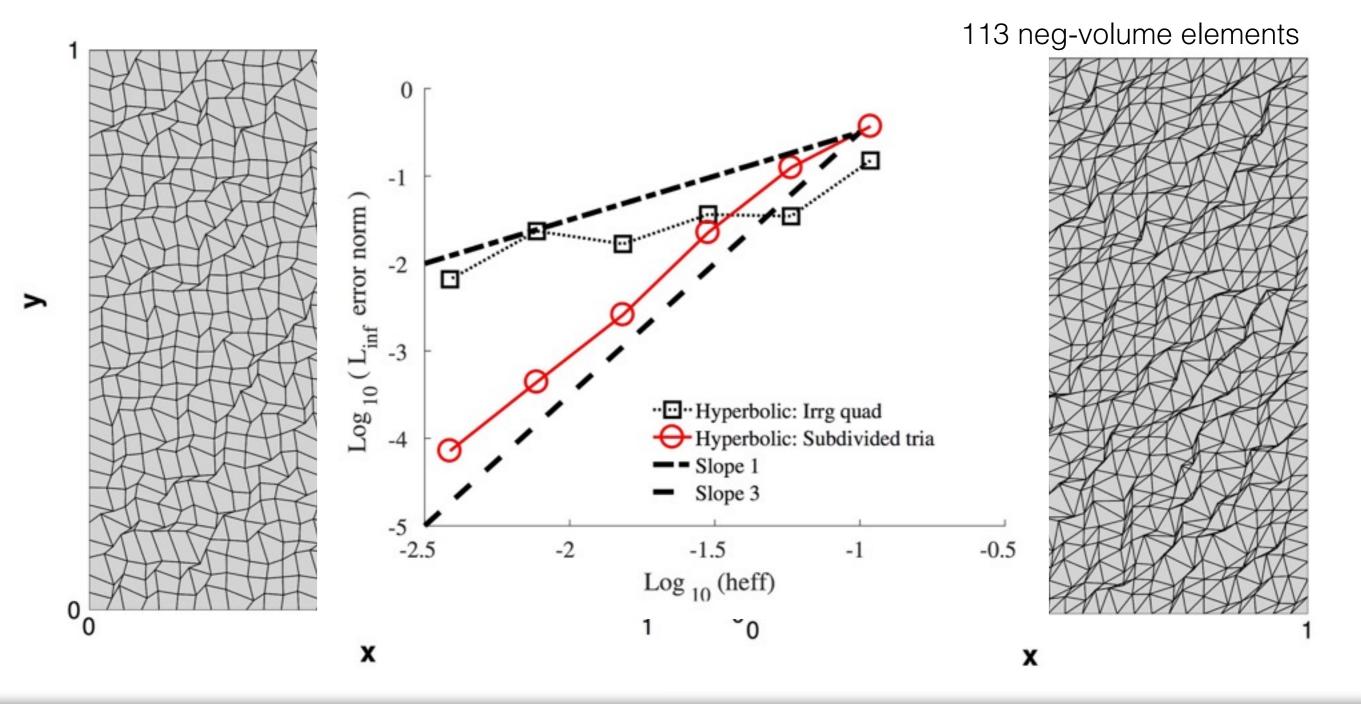
- Subdivided quads
- Discontinuity capturing
- Singularity
- Hanging nodes
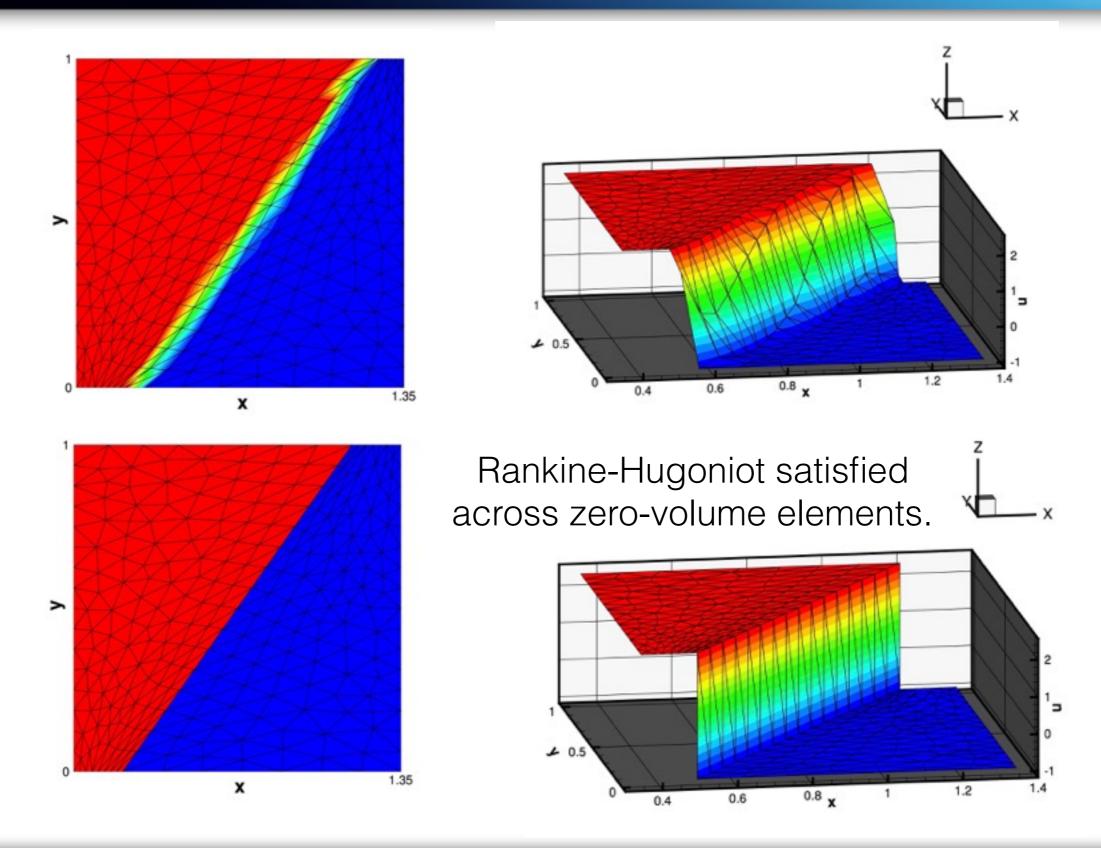- Overset grid

Subdivide Quads into Triangles

Subdivide quads into triangles to achieve 3rd-order accuracy. Negative volume elements may be generated, but that's OK.

113 neg-volume elements

Rankine-Hugoniot satisfied across zero-volume elements.

Reentrant corner problem: 1st-order accuracy at best [Laasonen 1967].
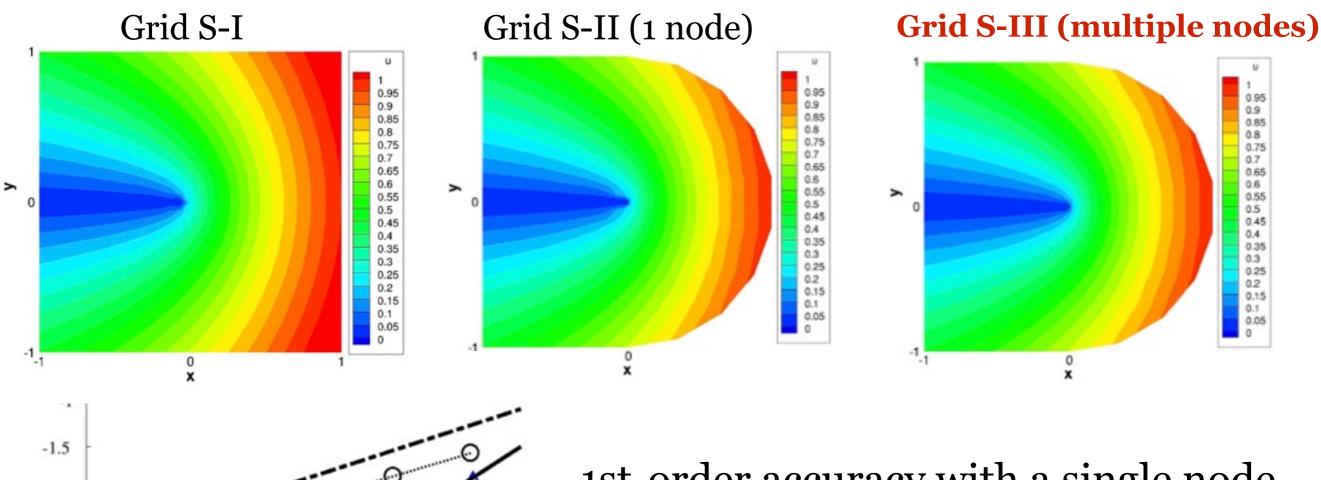
Place multiple nodes
at the origin to resolve
the singularity.

Singularity with Multiple Nodes
Laplace' equation

Grid S-I

Grid S-II (1 node)

Grid S-III (multiple nodes)

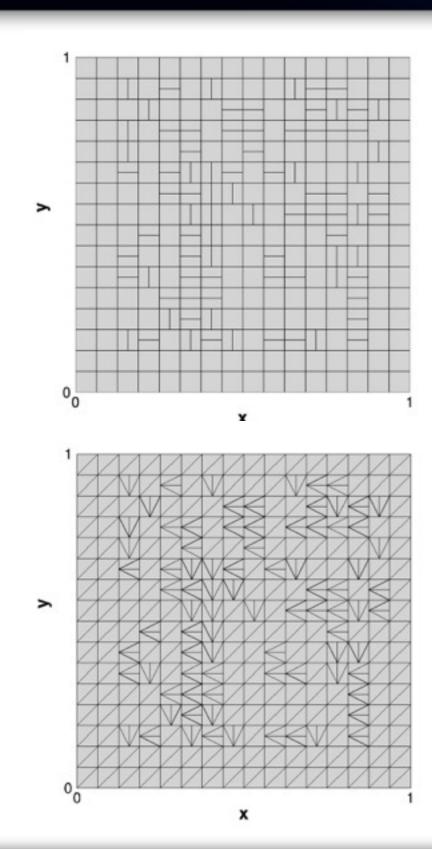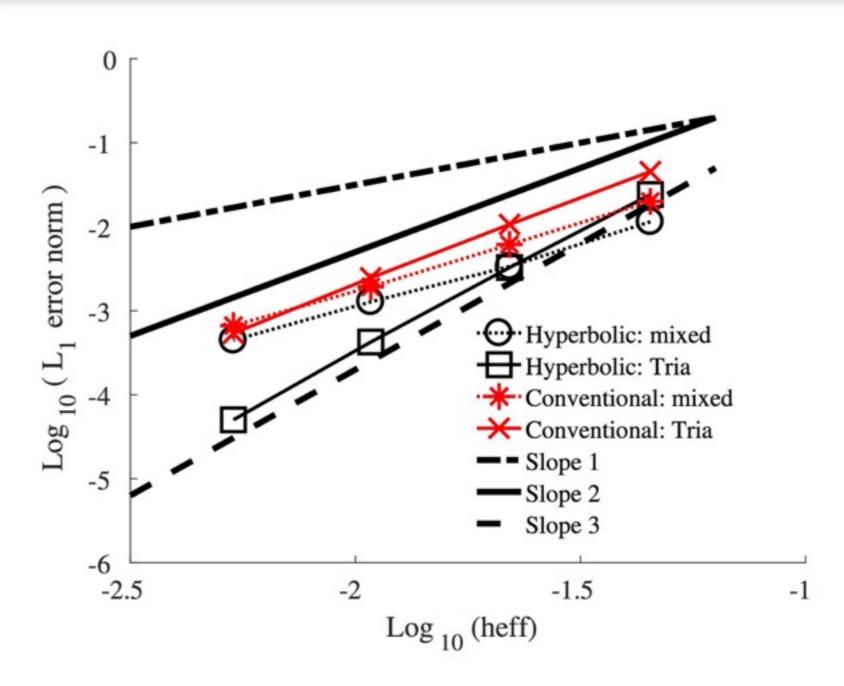1st-order accuracy with a single node at origin (expected).

2nd-order accuracy is achieved with multiple nodes at origin.

advection-diffusion equation

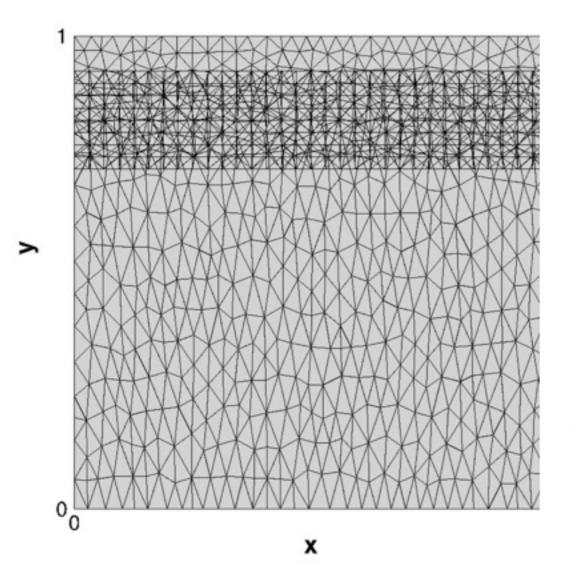EB scheme directly applied.
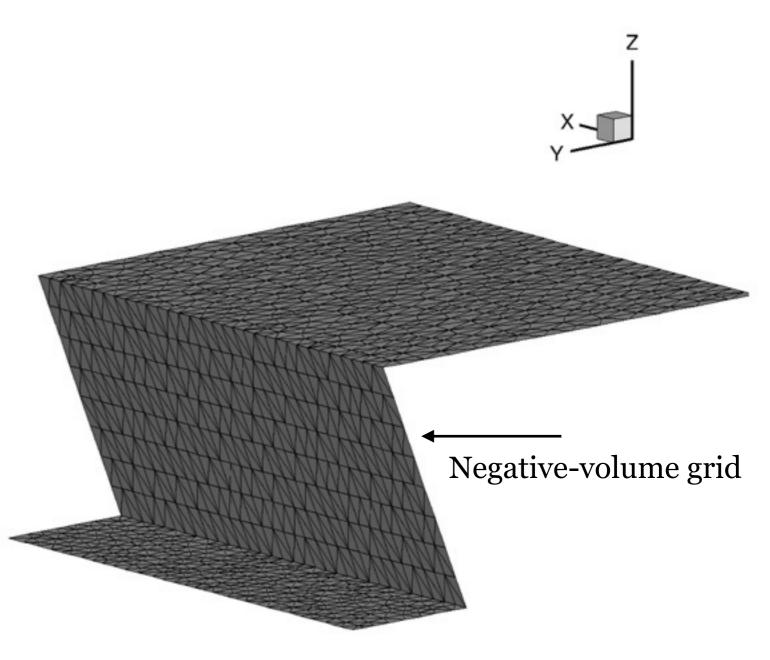3rd-order accuracy can be achieved.

Connect overset grids to create a single grid.

- Simplification in coding
- Discrete conservation

Negative-volume grid

As expected, 2nd-order accuracy by a conventional scheme, and 3rd-order accuracy by a hyperbolic scheme.

"Forbidden Capability"

Hiro demonstrated the usefulness of zero/negative-volume elements, and successfully published a conference paper. However, his task is not complete: theory, unsteady, Euler/Navier-Stokes, etc. His conference paper could turn out to be recognized as a landmark paper in future, or could be forgotten and varied in the shadow of the Twilight Zone.