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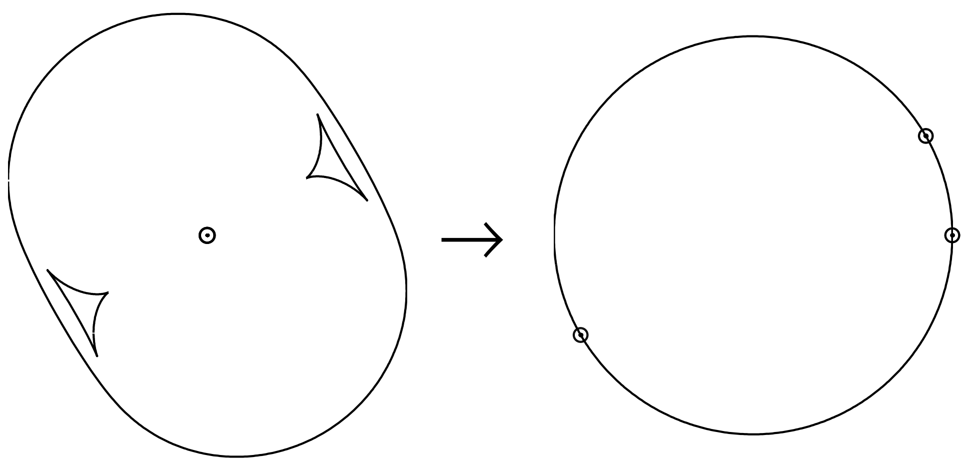

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A General Theory of Local Preconditioning and Its Application to the 2D Ideal MHD Equations

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Abstract

In this paper, we describe a general theory of local preconditioning which assigns a unique preconditioner for any set of two-dimensional hyperbolic partial differential equations. The preconditioner is optimal in the sense that it gives the lowest possible condition number (the ratio of maximum to minimum wave speeds). Taking the two-dimensional magnetohydrodynamic (MHD) equations as an example, we show how the preconditioner can be derived both approximately and exactly, and present some results including a numerical experiment to demonstrate its effectiveness.

Introduction

Local preconditioning is a technique to equalize as much as possible all the wave speeds of PDEs, so that the maximum local Courant number can be taken for all the waves (not just for the fastest one as in local time stepping)[6] and all the error modes propagate at the same rate thereby accelerating the convergence toward a steady state. A way to alter the transient solutions is to take a particular combination of the residual components to drive each solution component. This is done by multiplying the residual by a locally evaluated matrix (preconditioning matrix). Several such matrices have already been found and used in practice for the Euler or Navier-Stokes,[8,9] but a systematic procedure to derive a preconditioner for an arbitrary set of PDEs was not available. This makes it difficult to extend this very useful technique to other complicated but important PDEs such as the magnetohydrodynamic(MHD) equations.

In this paper, we report further development of the theory of local preconditioning previously presented at the 32nd AIAA Fluid Dynamics Conference 2002.[1] We now give a complete account of the general theory for constructing a unique local preconditioning matrix for any set of first-order PDEs in two dimensions. Although the theory gives very simple preconditioning matrices for some PDEs such as the Euler equations, it could yield in other cases rather complicated matrices not suitable for practical application. The ideal MHD equations, in particular, do lead us to such impractical results. Therefore, in this paper, we turn our attention to a numerical (and/or approximate) construction of the preconditioner. As to approximation, our focus is low-Mach-number flows where the condition number becomes singular, and also where the accuracy of compressible solvers may deteriorate as it happens to the Euler equations. Preconditioners are known to cure such problems.[3–5] Inspection of the wave pattern altered by a low-Mach-number MHD preconditioner shows that the low-speed singularity has been completely removed and all the wave speeds have successfully been made equal, giving the condition number = 1. Plots of the condition number over a range of Mach number are shown for the 2D MHD equations to demonstrate the effectiveness of the preconditioners derived from the theory. Also, we show that numerical implementation is also possible. Formulas necessary for numerical implementation are given.

The basic idea behind our construction of local preconditioning matrices is to decompose the PDE (based on its steady form) into a certain number of hyperbolic (advection) equations and/or a certain number of elliptic (Cauchy-Riemann) subsystems, and modify the wave speed of each subproblem independently to achieve as equal wave speeds as possible. In the next section, we describe the decomposition of the steady equations which gives the building blocks of the preconditioner. In Section 3, the form of the preconditioner is defined. In Section 4, the formulas for the acceleration factors are given, which completes the description of the construction of preconditioners. In Section 5, then, the MHD system is analyzed as an example. In Section 6 and 7, we present some useful formulas for implementing the preconditioner derived from the general theory. In Section 8, finally, we report numerical experiment for MHD nozzle flows to demonstrate the effectiveness of the preconditioner.

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Steady Characteristic Equations

Consider steady linear conservation laws of the form

$$\mathbf{A}\partial_x \mathbf{u} + \mathbf{B}\partial_y \mathbf{u} = 0 \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant and symmetric (the theory applies to nonlinear symmetrizable PDEs with appropriate linearization), and also we assume that $\mathbf{A}$ is invertible. Let λ_k and $\mathbf{r}_k$ be the k-th eigenvalue and the right eigenvector of the generalized eigenvalue problem

$$(\mathbf{B} - \lambda_k \mathbf{A})\mathbf{r}_k = 0. \tag{2}$$

By symmetry, $\mathbf{r}_k^T$ is the corresponding left eigenvector. Then, clearly, $\mathbf{r}_k$ is the right eigenvector of $\mathbf{A}^{-1}\mathbf{B}$. Also, $\mathbf{r}_k^T\mathbf{A}$ is the left eigenvector of $\mathbf{A}^{-1}\mathbf{B}$, and moreover it is orthogonal to the right eigenvectors: $\mathbf{r}_k^T\mathbf{A}\mathbf{r}_j = 0$ for $k \neq j$.

We obtain the k-th characteristic equation by multiplying (1) from the left by $\mathbf{r}_k^T$

$$\mathbf{r}_k^T(\mathbf{A}\partial_x \mathbf{u} + \mathbf{B}\partial_y \mathbf{u}) = 0 \tag{3}$$

$$\longrightarrow \quad \mathbf{r}_k^T\mathbf{A}(\partial_x \mathbf{u} + \lambda_k\, \partial_y \mathbf{u}) = 0 \tag{4}$$

$$\longrightarrow \quad \partial_x(R_k) + \lambda_k\, \partial_y(R_k) = 0 \tag{5}$$

where $R_k = \mathbf{r}_k^T\mathbf{A}\mathbf{u}$ is the Riemann invariant. If the eigenvalue is real, this is a scalar advection equation. If it is complex ($\lambda = \lambda_R + i\lambda_I$), then, by writing $R_k = R_R + iR_I$ and taking the real and the imaginary parts of (5), we obtain a 2x2 elliptic (Cauchy-Riemann) system

$$\mathbf{A}_e\partial_x(\mathbf{R}_e) + \mathbf{B}_e\partial_y(\mathbf{R}_e) = 0 \tag{6}$$

where

$$\mathbf{A}_e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}_e = \begin{bmatrix} \lambda_R & -\lambda_I \\ \lambda_I & \lambda_R \end{bmatrix} \tag{7}$$

$$\mathbf{R}_e = [R_R, R_I]^T = \mathbf{S}^T\mathbf{A}\mathbf{u} \tag{8}$$

Alternatively, this system can be obtained directly by

$$\mathbf{S}^T(\mathbf{A}\partial_x \mathbf{u} + \mathbf{B}\partial_y \mathbf{u}) = 0 \tag{9}$$

where $\mathbf{S} = [r_R, r_I]$.

Form of Preconditioner

Now consider the unsteady version of (1)

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} + \mathbf{B}\partial_y \mathbf{u} = 0. \tag{10}$$

Based on the characteristic decomposition developed in the previous section, we define the preconditioning matrix by

$$\mathbf{P} \equiv \sum_{\{k\}:\lambda_k\in\mathcal{R}} a_k \mathbf{r}_k \mathbf{r}_k^T + \sum_{\{k\}:\lambda_k\in\mathcal{C}} a_k \mathbf{S}_k \mathbf{S}_k^T \tag{11}$$

where a_k is a scalar factor (acceleration factor) that will be chosen to optimize the condition number. Note that the preconditioer is clearly symmetric. Then, the preconditioned system

$$\partial_t \mathbf{u} + \mathbf{P}[\mathbf{A}\partial_x \mathbf{u} + \mathbf{B}\partial_y \mathbf{u}] = 0 \tag{12}$$

can be written, by virtue of (5,9), as

$$\partial_t \mathbf{u} + \sum_{\{k\}:\lambda_k\in\mathcal{R}} a_k \mathbf{r}_k \{\partial_x(R_k) + \lambda_k\, \partial_y(R_k)\} + \sum_{\{k\}\lambda_k\in\mathcal{C}} a_k \mathbf{S}_k \{\mathbf{A}_e\partial_x(\mathbf{R}_e) + \mathbf{B}_e\partial_y(\mathbf{R}_e)\} = 0 \tag{13}$$

which shows clearly how the evolution of $\mathbf{u}$ is driven by each subproblem.

The idea behind the construction of the preconditioner in this form is that the matrix formed by the eigenvector, such as $\mathbf{r}_k\mathbf{r}_k^T$, projects the vector of PDEs (residual) onto the space of a one-dimensional subspace (or two-dimensional subspace if elliptic) spanned by the eigenvector and then the local wave speed for each subproblem is altered by assigning appropriate value to a_k to achieve an optimal condition number. Note that the projection is not precise at this point because of the ambiguity in the normalization of the eigenvectors. However, the effect of eigenvector normalization is only a matter of a scalar factor, which will emerge naturally in the acceleration factor, and therefore such ambiguity will be completely removed as we will see later.

In this paper, based on the steady characteristic decomposition, we refer to the second term in (13) as hyperbolic components and to the third as elliptic, although all are hyperbolic in time. It remains to determine the acceleration factors $\{a_k\}$.

Acceleration Factors

To optimize the condition number, we choose the acceleration factors $\{a_k\}$ such that the wave speeds of the subproblems are as nearly equal as possible. In the hyperbolic case, the subproblem can be easily obtained from (12) by using the left eigenvector.

$$\mathbf{r}_k^T\mathbf{A}\,\{\partial_t \mathbf{u} + \mathbf{P}[\mathbf{A}\partial_x \mathbf{u} + \mathbf{B}\partial_y \mathbf{u}]\} = 0 \tag{14}$$

which, by orthogonality, becomes

$$\partial_t(R_k) + a_k(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)\,\{\partial_x(R_k) + \lambda_k\, \partial_y(R_k)\} = 0. \tag{15}$$

This shows that the Riemann invariant R_k propagates at the speed

$$a_k|\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k|\sqrt{1+\lambda_k^2}. \tag{16}$$

Suppose we wish to make it propagate at a speed C (typically taken to be the local Mach number). We can achieve this by setting

$$a_k = \frac{C}{|\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k|\sqrt{1+\lambda_k^2}}. \tag{17}$$

Note that substituting this into (13) gives an equation that suffers no arbitrary scaling of eigenvectors (the factor $|\mathbf{r}_k^T \mathbf{A} \mathbf{r}_k|$ normalizes the corresponding eigenvector). In this way, wave speeds for all the hyperbolic subproblems can be made equal, implying perfect preconditioing for PDEs whose steady characteristics are all hyperbolic.

If there are elliptic components, each elliptic subsystem can be extracted similarly by multiplication of the vector, $\mathbf{S}^T \mathbf{A}$. Again by orthogonality, we obtain

$$\partial_t(\mathbf{R}_e) + a_k \mathbf{S}^T \mathbf{A} \mathbf{S} \left\{ \mathbf{A}_e \partial_x(\mathbf{R}_e) + \mathbf{B}_e \partial_y(\mathbf{R}_e) \right\} = 0. \tag{18}$$

As will be shown in the next section, the domain of influence of this elliptic system is an ellipse. The length of the major axis of this ellipse, which is the maximum wave speed for this system, is given by

$$\frac{a_k |\lambda_I|}{k_e \sqrt{\frac{1}{2}(1 + \lambda_R^2 + \lambda_I^2 - R)}} \tag{19}$$

where

$$R = \sqrt{(1 - \lambda_R^2 - \lambda_I^2)^2 + 4\lambda_R^2} \tag{20}$$

$$k_e = \frac{1}{\sqrt{(\mathbf{r}_R^T \mathbf{A} \mathbf{r}_R)^2 + (\mathbf{r}_R^T \mathbf{A} \mathbf{r}_I)^2}}. \tag{21}$$

We can therefore make the maximum speed be C by setting

$$a_k = \frac{k_e C \sqrt{\frac{1}{2}(1 + \lambda_R^2 + \lambda_I^2 - R)}}{|\lambda_I|}. \tag{22}$$

Note that the factor k_e gives the correct normalization factor for the eigenvector $\mathbf{S}$ (normalization in its complex form), and therefore the final form of the equation again does not suffer from arbitrary scaling. For elliptic subsystems, there are no degrees of freedom left to adjust the minimum wave speed. It can be shown that any further preconditioning does not improve its condition number (the ratio of the major axis to the minor). Therefore, perfect preconditioning is not possible unless the domain of influence happens to be a circle.

Elliptic Subsystem

Consider the 2x2 elliptic subsystem.

$$\partial_t(\mathbf{R}_e) + a_k \mathbf{S}^T \mathbf{A} \mathbf{S} \left\{ \mathbf{A}_e \partial_x(\mathbf{R}_e) + \mathbf{B}_e \partial_y(\mathbf{R}_e) \right\} = 0. \tag{23}$$

Orthogonality of the eigenvectors will allow slight simplification of the system. By orthogonality, we have

$$(\mathbf{r}_R^T + i\mathbf{r}_I^T)\mathbf{A}(\mathbf{r}_R^T - i\mathbf{r}_I^T) = 0 \tag{24}$$

which yields

$$k_1 \equiv \mathbf{r}_R^T \mathbf{A} \mathbf{r}_R = -\mathbf{r}_I^T \mathbf{A} \mathbf{r}_I \tag{25}$$

$$k_2 \equiv \mathbf{r}_R^T \mathbf{A} \mathbf{r}_I = \mathbf{r}_I^T \mathbf{A} \mathbf{r}_R. \tag{26}$$

Therefore, we have

$$\mathbf{S}^T \mathbf{A} \mathbf{S} = \begin{bmatrix} \mathbf{r}_R^T \mathbf{A} \mathbf{r}_R & \mathbf{r}_R^T \mathbf{A} \mathbf{r}_I \\ \mathbf{r}_I^T \mathbf{A} \mathbf{r}_R & \mathbf{r}_I^T \mathbf{A} \mathbf{r}_I \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ k_2 & -k_1 \end{bmatrix}. \tag{27}$$

The system (23) then becomes

$$\partial_t(\mathbf{R}_e) + a_k \begin{bmatrix} k_1 & k_2 \\ k_2 & -k_1 \end{bmatrix} \partial_x(\mathbf{R}_e) \tag{28}$$

$$+ a_k \begin{bmatrix} k_1\lambda_R + k_2\lambda_I & -k_1\lambda_I + k_2\lambda_R \\ k_2\lambda_R - k_1\lambda_I & -k_2\lambda_I - k_1\lambda_R \end{bmatrix} \partial_y(\mathbf{R}_e) = 0. \tag{29}$$

This has plane wave solutions $\mathbf{R}_e = \mathbf{f}(Vt - x\cos\theta - y\sin\theta)$ if

$$V^2 = a_k^2(k_1^2 + k_2^2)\{(\cos\theta + \lambda_R \sin\theta)^2 + \lambda_I^2 \sin^2\theta\}, \tag{30}$$

and it can be shown that the domain of influence is given by

$$(\lambda_R^2 + \lambda_I^2)x^2 - 2\lambda_R xy + y^2 = a_k^2(k_1^2 + k_2^2)\lambda_I^2. \tag{31}$$

We write this equation as

$$\mathbf{z}^T \mathbf{F}\, \mathbf{z} = 1 \tag{32}$$

where $\mathbf{z} = [x, y]^T$ and

$$\mathbf{F} = \frac{1}{a_k^2(k_1^2 + k_2^2)\lambda_I^2} \begin{bmatrix} \lambda_R^2 + \lambda_I^2 & -\lambda_R \\ -\lambda_R & 1 \end{bmatrix}. \tag{33}$$

Note that the matrix $\mathbf{F}$ is *positive definite* provided $\lambda_I \neq 0$. Therefore, the domain of influence is an ellipse. Moreover it is centered at the origin. Let λ_S denote the smaller of the two eigenvalues of $\mathbf{F}$. Then, the length of the major axis is given by

$$\frac{1}{\sqrt{\lambda_S}} = \sqrt{\frac{2\, a_k^2(k_1^2 + k_2^2)\lambda_I^2}{(1 + \lambda_R^2 + \lambda_I^2) - R}} = \frac{a_k |\lambda_I| \sqrt{k_1^2 + k_2^2}}{\sqrt{\frac{1}{2}(1 + \lambda_R^2 + \lambda_I^2 - R)}} \tag{34}$$

which is the maximum wave speed. The minimum wave speed is given similarly by the larger eigenvalue, λ_L. The condition number is the ratio of the two and found to be

$$\left(\frac{1}{\sqrt{\lambda_S}}\right) / \left(\frac{1}{\sqrt{\lambda_L}}\right) = \frac{\sqrt{\lambda_L}}{\sqrt{\lambda_S}} = \sqrt{\frac{1 + \lambda_R^2 + \lambda_I^2 + R}{1 + \lambda_R^2 + \lambda_I^2 - R}}. \tag{35}$$

This is the optimal condition number of the elliptic system because any further preconditioning does not improve it (the proof is omitted here, and will be reported elsewhere). Also, this is the optimal condition number of the whole system as well because any hyperbolic component can be preconditioned perfectly.

2D Ideal MHD Equations in the Low-Mach-Number Limit

The equations governing linearized two-dimensional magnetohydrodynamic flows are of the form (10) with variables

$$\mathbf{u} = \left(\frac{p}{\rho_0 a_0^2}, \frac{u}{a_0}, \frac{v}{a_0}, \frac{B_x}{\sqrt{\rho_0} a_0}, \frac{B_y}{\sqrt{\rho_0} a_0}, \frac{p}{\rho_0 a_0^2} - \frac{\rho}{\rho_0}\right)^T, \tag{36}$$

$$\mathbf{A} = \begin{bmatrix} M & 1 & 0 & 0 & 0 & 0 \\ 1 & M & 0 & 0 & b\sin\alpha & 0 \\ 0 & 0 & M & 0 & -b\cos\alpha & 0 \\ 0 & 0 & 0 & M & 0 & 0 \\ 0 & b\sin\alpha & -b\cos\alpha & 0 & M & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \tag{37}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b\sin\alpha & 0 & 0 \\ 1 & 0 & 0 & b\cos\alpha & 0 & 0 \\ 0 & -b\sin\alpha & b\cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

where M is q/a, b is the magnitude of the magnetic field, and α is the angle between the streamline and the magnetic field. In,[1] we focused on the 'aligned flow' case where $\alpha = 0$. This gives rise to a preconditioner that can be expressed in relatively simple closed form. Here, we remove that restriction, but remain in two-dimensions. To construct the preconditioning matrix, we need to find the eigenvalues of the matrix $\mathbf{A}^{-1}\mathbf{B}$. Two of them are zero with the eigenvectors;

$$\mathbf{r}_1 = [0,0,0,0,0,1]^T, \quad \mathbf{r}_2 = [0,0,0,0,1,0]^T \tag{39}$$

which correspond to the entropy wave and the divergence wave respectively. The remaining four are the roots of the quartic equation;

$$(1+\lambda^2)\left\{(\lambda\cos\alpha - \sin\alpha)^2 - (1+b^2)M^2\lambda^2\right\} = -M^4\lambda^4. \tag{40}$$

This can be solved in principle, but the results are too complicated to be useful. For small Mach numbers, good approximations are

$$\lambda_{3,4} = \pm i + \mathcal{O}(M^4) \tag{41}$$

$$\lambda_{5,6} = \frac{b\sin\alpha}{b\cos\alpha \pm M\sqrt{1+b^2}} + \mathcal{O}(M^3) \tag{42}$$

where $i = \sqrt{-1}$ and therefore the fast waves are elliptic (the unsteady waves travel omnidirectionally). The approximations given above are valid for arbitrary values of b and α, so that we can reach the hydrodynamic limit as B goes to zero. The expression for the corresponding eigenvectors can be obtained by substituting the above values to the vector given below

$$\mathbf{r} = \begin{bmatrix} b^2\sin\alpha(\lambda\cos\alpha - \sin\alpha)(1+\lambda^2) + M^2\lambda^2 \\ -Mb^2\lambda\cos\alpha\sin\alpha(1+\lambda^2) - M\lambda^2 \\ -Mb^2\lambda\sin^2\alpha(1+\lambda^2) - M((1-M^2)\lambda^3 \\ -b\lambda(\lambda\cos\alpha - \sin\alpha) + M^2\lambda^2 b\cos\alpha \\ -b\lambda^2(\lambda\cos\alpha - \sin\alpha) + M^2\lambda^3 b\cos\alpha \\ 0 \end{bmatrix} \tag{43}$$

which is exact provided λ is exact. For the fast wave ($\lambda_{3,4}$), by substitution, we obtain

$$\mathbf{r}_{3,4} = \begin{bmatrix} M^2 \\ -M \\ \pm iM \\ -b\cos\alpha \pm i\, b\sin\alpha \\ b\sin\alpha \pm i\, b\cos\alpha \\ 0 \end{bmatrix} \tag{44}$$

so that we may take

$$\mathbf{r}_R = \begin{bmatrix} M^2 \\ -M \\ 0 \\ -b\cos\alpha \\ b\sin\alpha \\ 0 \end{bmatrix}, \quad \mathbf{r}_I = \begin{bmatrix} 0 \\ 0 \\ -M \\ -b\sin\alpha \\ -b\cos\alpha \\ 0 \end{bmatrix} \tag{45}$$

where some of the terms of $\mathcal{O}(M^2)$ have been neglected, nevertheless the approximation is still valid for arbitrary values of b and α. Similarly for the slow waves ($\lambda_{5,6}$), we obtain

$$\mathbf{r}_{5,6} = \begin{bmatrix} B^2 \\ M \pm b\cos\alpha\sqrt{1+b^2} \\ \pm b\sin\alpha\sqrt{1+b^2} \\ -b\cos\alpha \mp M\sqrt{1+b^2} \\ -b\sin\alpha \\ 0 \end{bmatrix}. \tag{46}$$

The preconditioning matrix can now be constructed by assembling the results,

$$\mathbf{P} = \sum_{k=1,2,5,6} a_k \mathbf{r}_k \mathbf{r}_k^T + a_3(\mathbf{r}_R\mathbf{r}_R^T + \mathbf{r}_I\mathbf{r}_I^T)_3 \tag{47}$$

where the common wave speed C is taken to be the local Mach number.

Since we have the exact expression for the eigenvectors, we can construct a preconditioner without any approximation provided the eigenvalues are available. This can be done numerically as will be described in the next section.

Figures 1 and 2 show the effect of preconditioning. Figure 1 shows the wave diagram of the original MHD equations for the parameters $M = 0.001$, $B = 0.9$, and $\alpha = 20°$. The condition number is 1.3457E+03. Figure 2 shows the wave diagram of the preconditioned MHD equations, with the low-Mach-number preconditioer, for the same parameters. Note that each of the

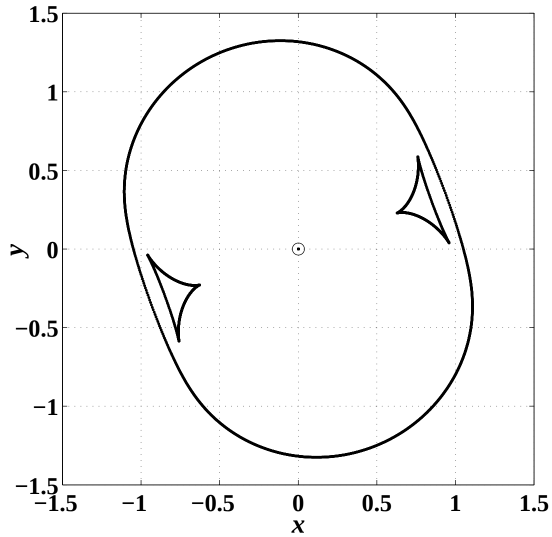

Fig. 1 Wave diagram for the original MHD equations

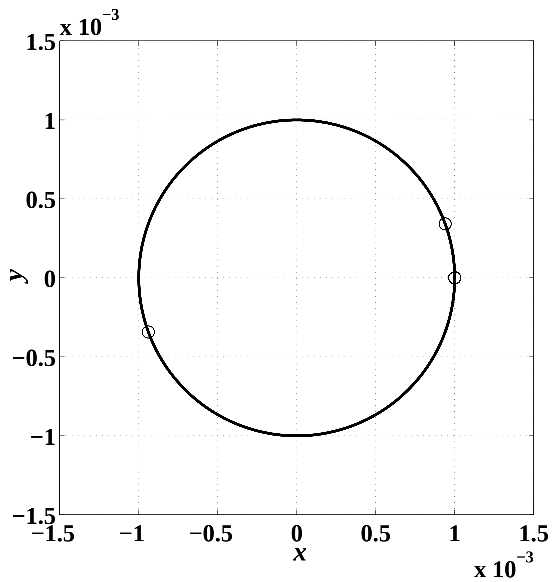

Fig. 2 Wave diagram for the preconditioned MHD equations

slow waves has been made simple advection (indicated by the circles off the x-axis, and the entropy and divergence waves are indicated by the circle on the x-axis), and that all the waves now travel at the same speed (i.e. Mach number). The condition number has successfully been made 1.0000E+00, demonstrating huge improvement over the original system.

Figure 3, 4, and 5 show the variation of the condition number against the Mach number for $b = 0.1$ and $\alpha = 30°$ The original MHD equations suffer from a singularity at low-Mach-number as seen in Figure 3. This has been completely removed in the preconditioned system with either the approximate or the (numerically constructed) exact preconditioner (See Figures 4 and 5). Note also that the singularity at $M = 1$

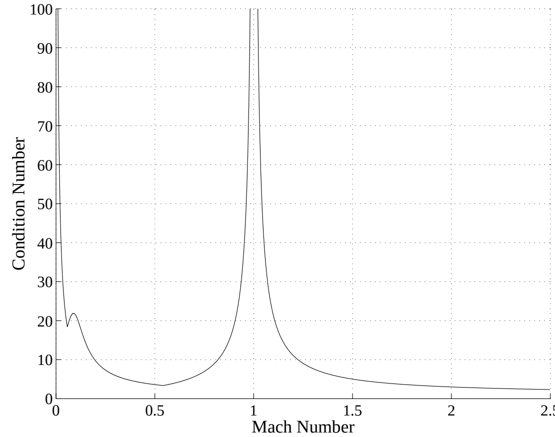

Fig. 3 Condition number for the original MHD equations

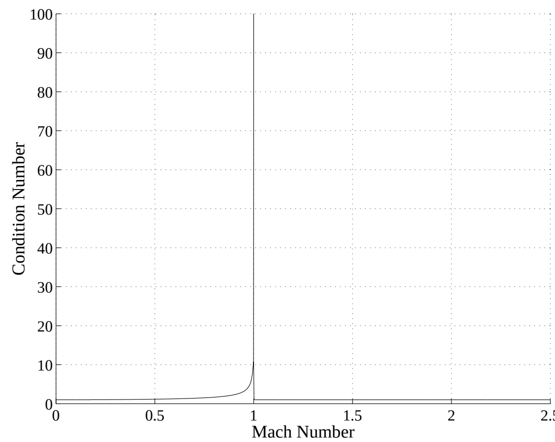

Fig. 4 Condition number for the MHD equations preconditioned with the exact preconditioner.

has been greatly weakened. Note that the low-Mach-number preconditioner works remarkably well over a wider range of Mach number than anticipated. The condition number is greatly reduced approximately up to $M = 1.8$. We see some wiggling behavior at low Mach number in the case of the approximate preconditioner. This is caused by a singularity of the eigenvalue (42): $b\,\cos\alpha$ is very close to $M\sqrt{1+b^2}$. Some fix is required to suppress this behavior. The exact preconditioner gives the lowest condition number, realizing the perfect conditioning (See Figure 6) except in the range approximately $0.4 < M < 1.0$ where the fast wave mode is expected to become elliptic with the domain of influence of ellipse (not a circle as in the low-Mach-number limit).

Solving A Quartic Equation

If it is preferred to implement the preconditioner without any approximation, it can be done numerically. To this end, we would need the eigenvalues and the eigenvectors. The eigenvectors can be computed by (43) once the eigenvalues are known. A difficulty is to numerically compute the eigenvalues: the solutions

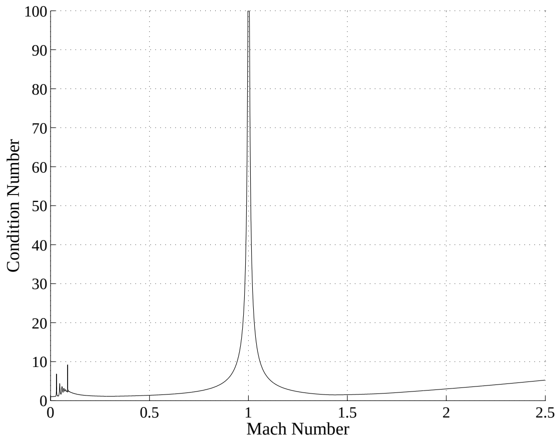

Fig. 5 Condition number for the MHD equations preconditioned with the approximate preconditioner.

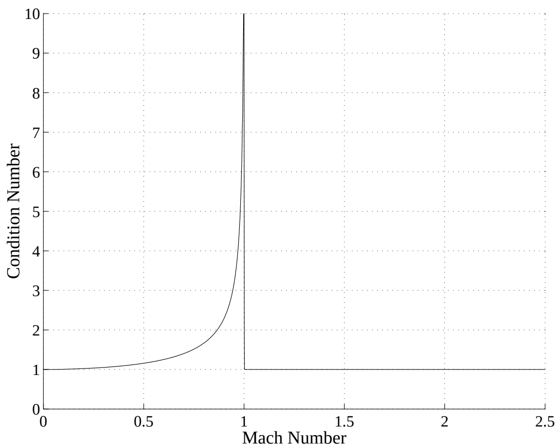

Fig. 6 Blow-up of Figure 4. Vertical axis now runs from zero to 10.

to the quartic (40). Although there exists an analytical formula for quartic equations, it is not only too expensive for practical purposes but also difficult to implement in a stable way. For this reason, we have developed a robust and efficient algorithm to solve the quartic, based on Newton's method.

Consider a quartic equation

$$a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \qquad (48)$$

We assume that there exist two real solutions to the quartic, which applies to the 2D MHD equations (slow wave). We begin by computing one of the two real solutions by Newton's method with the initial guess taken to be the low-Mach-number approximation

$$\lambda_0 = \frac{b\,\sin\alpha}{(b\,\cos\alpha + M\sqrt{1+b^2}) - h} \qquad (49)$$

where

$$h = \frac{M^3 b^2 \sin^2\alpha}{2\sqrt{1+b^2}\left\{b^2\sin^2\alpha + (b\,\cos\alpha + M\sqrt{1+b^2})^2\right\}}. \qquad (50)$$

The term h upgrades the approximation (42) to the sixth order accurate one in Mach number $\mathcal{O}(M^6)$. This is sufficiently accurate for small Mach numbers and also a good initial guess for large Mach numbers. It has been confirmed that only a few Newton iterations are required to obtain the solution comparable to the one obtained by the exact formula.

The second step is to find another real root λ_2. We use Newton's method again, but we do not solve the quartic because it may converge to the same root λ_1 that we have already found. In order to avoid such a possibility, we apply the method to the cubic equation

$$\begin{aligned} a_4\lambda^3 \quad &+ \quad (a_3+\lambda_1 a_4)\lambda^2 + \{a_2 + \lambda_1(a_3+\lambda_1 a_4)\}\,\lambda \\ &+ \quad [a_1 + \lambda_1\{a_2+\lambda_1(a_3+\lambda_1 a_4)\}] = 0 \end{aligned} \qquad (51)$$

which is derived from dividing (48) by $(\lambda - \lambda_1)$, thus excluding the root already found. The initial guess is the other low-Mach-number approximation

$$\lambda_0 = \frac{b\,\sin\alpha}{(b\,\cos\alpha - M\sqrt{1+b^2}) + h}. \qquad (52)$$

This Newton iteration has also been found to converge for only a few iterations.

Now that we have found two real roots, λ_1, λ_2, we only need to solve a quadratic equation to find two remaining roots. The quadratic equation can be found by dividing (51) by $(\lambda - \lambda_2)$. The result is

$$a_4\lambda^2 + (a_3+\alpha a_4)\lambda + \left\{a_2 + \alpha a_3 + (\alpha^2-\beta)a_4\right\} = 0, \qquad (53)$$

where

$$\alpha = \lambda_1 + \lambda_2, \quad \beta = \lambda_1\lambda_2, \qquad (54)$$

which is easily solved by the quadratic formula. This algorithm is not only much faster than the exact formula (5 times faster in CPU time on average), but also robust. we have found also that this algorithm is about 25 times faster than the general purpose subroutine (based on Laguerre's method) available in.[12] The method can be generalized easily for solving arbitrary quartic equations by extending Newton's method to compute complex solutions provided good initial solutions for two of the four roots are available.

Modified Flux Function

It is well-known that the combination of a preconditioner and the standard Roe flux function suffers from a severe stability restriction.[7] In order to remove the restriction, the dissipation term in the flux function must be formulated based on the preconditioned system rather than the original one. For the preconditioner derived from our theory, this task is straightforward because the wave structure of the preconditioned system is clear. The formulas necessary to implement this will be given below in the general form.

Consider a 2D linearized symmetrizable conservation law in the conservative form,

$$\mathbf{U}^c{}_t + \mathbf{A}^c\mathbf{U}^c_x + \mathbf{B}^c\mathbf{U}^c_y = 0. \tag{55}$$

This is transformed into a symmetric form by a transformation matrix $\mathbf{T}$,

$$\mathbf{T}(\mathbf{U}^c{}_t + \mathbf{A}^c\,\mathbf{U}^c_x + \mathbf{B}^c\,\mathbf{U}^c_y) = 0 \tag{56}$$

$$\longrightarrow \quad \mathbf{U}_t + \mathbf{A}^s\mathbf{U}_x + \mathbf{B}^s\mathbf{U}_y = 0 \tag{57}$$

where

$$\mathbf{A}^s = \mathbf{T}\mathbf{A}^c\mathbf{T}^{-1} \quad \mathbf{B}^s = \mathbf{T}\mathbf{B}^c\mathbf{T}^{-1}. \tag{58}$$

The symmetric form may be written in the streamline coordinates (s, n),

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_s + \mathbf{B}\mathbf{U}_n = 0 \tag{59}$$

where

$$\mathbf{A} = \mathbf{A}^s\cos\theta + \mathbf{B}^s\sin\theta, \quad \mathbf{B} = \mathbf{B}^s\cos\theta - \mathbf{A}^s\sin\theta \tag{60}$$

and θ is the flow angle.

Note that if $\mathbf{P}$ denotes a symmetric preconditioner to be applied to the symmetric form

$$\mathbf{U}_t + \mathbf{P}(\mathbf{A}\mathbf{U}_s + \mathbf{B}\mathbf{U}_n) = 0, \tag{61}$$

then the corresponding preconditioner for the conservative form to be used as

$$\mathbf{U}^c_t + \mathbf{P}_c(\mathbf{A}^c\mathbf{U}^c_x + \mathbf{B}^c\mathbf{U}^c_y) = 0 \tag{62}$$

is given by

$$\mathbf{P}_c = \mathbf{T}^{-1}\mathbf{P}\mathbf{T}. \tag{63}$$

The Roe scheme is formulated using the Jacobian normal to a cell face, $\vec{\mathbf{n}} = (n_x, n_y) = (\cos\phi, \sin\phi)$,

$$\mathbf{A}^c_n = \mathbf{A}^c\,\cos\phi + \mathbf{B}^c\,\sin\phi, \tag{64}$$

giving the numerical flux

$$\frac{1}{2}(\mathbf{F}_R + \mathbf{F}_L) - \frac{1}{2}|\mathbf{A}^c_n|\Delta\mathbf{U}^c. \tag{65}$$

The dissipation term must be formulated based on the preconditioned system, and it is modified as follows.

$$\frac{1}{2}(\mathbf{F}_R + \mathbf{F}_L) - \frac{1}{2}\mathbf{P}_c^{-1}|\mathbf{P}_c\mathbf{A}^c_n|\Delta\mathbf{U}^c \tag{66}$$

where $\mathbf{P}_c^{-1}$ has been introduced to cancel the preconditioner that is to be multiplied from the left. Since we know exactly the wave structure of the preconditioned system (the eigenvalues and the eigenvectors), the term $|\mathbf{P}_c\mathbf{A}^c_n|$ can be written out relatively easily. The result is

$$\frac{1}{2}(\mathbf{F}_R + \mathbf{F}_L) - \frac{1}{2}\mathbf{P}_c^{-1}\left\{\sum_{hyperbolic}|\lambda^\star_k|\,\alpha^\star_k\;\mathbf{r}^\star_k \right.$$
$$\left. + \sum_{elliptic}\left(|\lambda^+_e|\,\alpha^+_e\;\mathbf{r}^+_e + |\lambda^-_e|\,\alpha^-_e\;\mathbf{r}^-_e\right)\right\} \tag{67}$$

where

$$\lambda^\star_k = \frac{C\left\{\cos(\phi-\theta) + \lambda_k\sin(\phi-\theta)\right\}}{\sqrt{1+\lambda_k^2}} \tag{68}$$

$$\alpha^\star_k = \frac{\mathbf{r}_k^T\mathbf{A}}{|\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k|}\mathbf{T}\,\Delta\mathbf{U} \tag{69}$$

$$\mathbf{r}^\star_k = \mathbf{T}^{-1}\mathbf{r}_k \tag{70}$$

$$\lambda^\pm_e = \pm\frac{a_e}{k_e}\sqrt{w_R^2 + w_I^2} \tag{71}$$

$$\alpha^\pm_e = \frac{\mathbf{r}^{\pm\,T}_{en}}{|\mathbf{r}^{\pm\,T}_{en}\,\mathbf{r}^\pm_{en}|}\mathbf{S}^T\mathbf{A}\mathbf{T}\,\Delta\mathbf{U} \tag{72}$$

$$\mathbf{r}^\pm_e = \mathbf{T}^{-1}\mathbf{S}(\mathbf{S}^T\mathbf{A}\mathbf{S})^{-1}\,\mathbf{r}^\pm_{en} \tag{73}$$

$$\mathbf{r}^\pm_{en} = \begin{bmatrix} k_2 w_I + k_1 w_R + \lambda^\pm_e \\ k_1 w_I - k_2 w_R \end{bmatrix} \tag{74}$$

$$w_R = \cos(\phi-\theta) + \lambda_R\sin(\phi-\theta) \tag{75}$$

$$w_I = \sin(\phi-\theta)\lambda_I. \tag{76}$$

For simple PDEs, this formula can be used to obtain analytical expression for the modified dissipation term. For complicated systems, this can be implemented numerically.

Results

The preconditioner has been tested for a two-dimensional nozzle flow. The shape of the nozzle is given by a simple cosine curve.

$$y = 0.15\cos\left(\frac{\pi}{3}x\right) + 0.35. \tag{77}$$

Figure 7 shows the grid used in the computation. Because of the symmetry, we use only the upper half of the nozzle. The solution we seek is a smooth flow that starts with low-Mach-number subsonic flow and accelerates through the nozzle toward supersonic flow.

A standard finite-volume discretization is employed with Van Leer's κ−scheme[10] with $\kappa = 0$ (without limiting). Note that solutions will not be monotonic near shocks for this choice of κ; to the test problem used in this work this does not apply. To reach a steady state we integrate the discretized system in time using the 4-stage time-stepping scheme developed for $\kappa = 0$ by Lynn and Van Leer.[11] As the boundary conditions, at the inlet (the left end), we prescribe in a column of ghost cells the following values:

$$M = 0.240 \tag{78}$$

$$v = 0.0 \tag{79}$$

$$p = 0.961 \tag{80}$$

$$s = 0.0 \tag{81}$$

$$b = 0.1 \tag{82}$$

$$\alpha = 0.0 \tag{83}$$

where s denotes the entropy, and let the Riemann solver find the inlet fluxes. On the top and the bottom

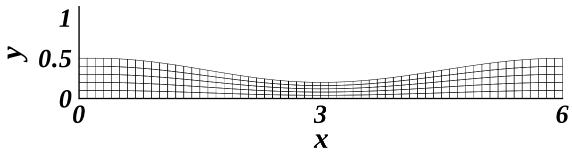

Fig. 7 The grid used for the computation.

(symmetry line) boundaries, we impose a reflection boundary condition (no normal velocity). At the exit (the right end), we give a no-gradient boundary condition: copy the interior state to the ghost cell and again let the Riemann solver find the flux. As the exit flow becomes supersonic, this automatically produces the upwind fluxes.

For this test problem, the magnetic field is nearly aligned with the flow direction throughout. Therefore, we have chosen to implement the preconditioner for aligned flows we derived in the previous work.[1] For the sake of completeness, the eigenvalues and eigenvectors are shown below.

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0 \tag{84}$$

$$\lambda_{5,6} = \pm\sqrt{\frac{M^2(1+b^2)-b^2}{(M^2-1)(M^2-b^2)}} \tag{85}$$

$$\mathbf{r}_1 = (0,0,0,0,0,1)^T \tag{86}$$

$$\mathbf{r}_2 = (0,0,0,0,1,0)^T \tag{87}$$

$$\mathbf{r}_{3,4} = \left(\frac{-b}{\sqrt{1+b^2}}, \pm 1, 0, \frac{1}{\sqrt{1+b^2}}, 0, 0\right)^T \tag{88}$$

$$\mathbf{r}_{5,6} = \left(M, -1, \lambda\beta^2, \frac{\beta^2 b}{M}, \frac{\lambda\beta^2 b}{M}, 0\right)^T \tag{89}$$

where $\beta^2 = M^2 - 1$. We implemented the preconditioner numerically using these eigenvalues and eigenvectors, using the modified Roe scheme described in the previous section.

Convergence history is shown in Figure 8. The vertical axis is the component of the residual corresponding to the density, scaled by the initial value of the residual. The horizontal axis is the number of 4-stage time steps (work unit). Clearly, preconditioning accelerates the convergence, reducing the number of time steps to reach the steady state by a factor 4.

Concluding Remarks

A complete account of the general theory of preconditioning has been presented. Taking the two-dimensional MHD system as an example, we have shown that the resulting preconditioner (either approximate or exact) greatly reduces the condition number of the system. This in turn has been confirmed by a numerical experiment. It was also shown that the preconditioners completely remove the low-Mach-number singularity in the case of MHD.

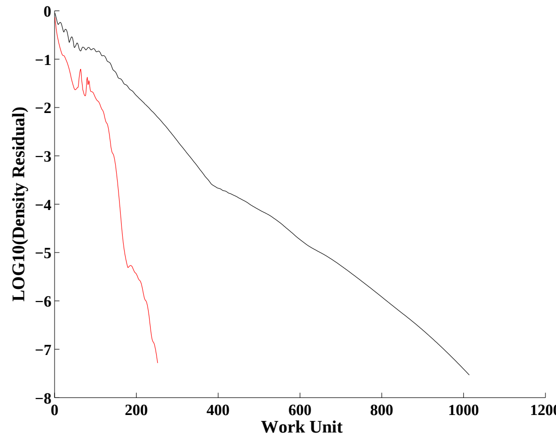

Fig. 8 Convergence History (Density residual vs The number of time steps)

Future work will include further analysis of the low-Mach-number preconditioner. First, the fix for the eigenvalue singularity for the slow wave must be developed. Second, because this approximate preconditioner works remarkably over a wide range of parameters, we may expect that the whole matrix may be simplified further without losing its effectiveness. Further numerical experiments would be desirable to numerically demonstrate the faster convergence over a range of parameters.

Extension to three-dimensional PDEs remains a challenge. One direction would be to intelligently guess the three-dimensional version from the two-dimensional matrix. For this purpose, further simplification to an approximate preconditioner may hold a key.

Finally, this work has paved the way for the ultimate $\mathcal{O}(N)$ convergence for MHD equations, by providing one of the building blocks for the textbook multigrid convergence, which we demonstrated previously for the Euler equations.[2]

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