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Face-Averaged Nodal-Gradient Approach to Cell-Centered Finite-Volume Method on Triangular Grids

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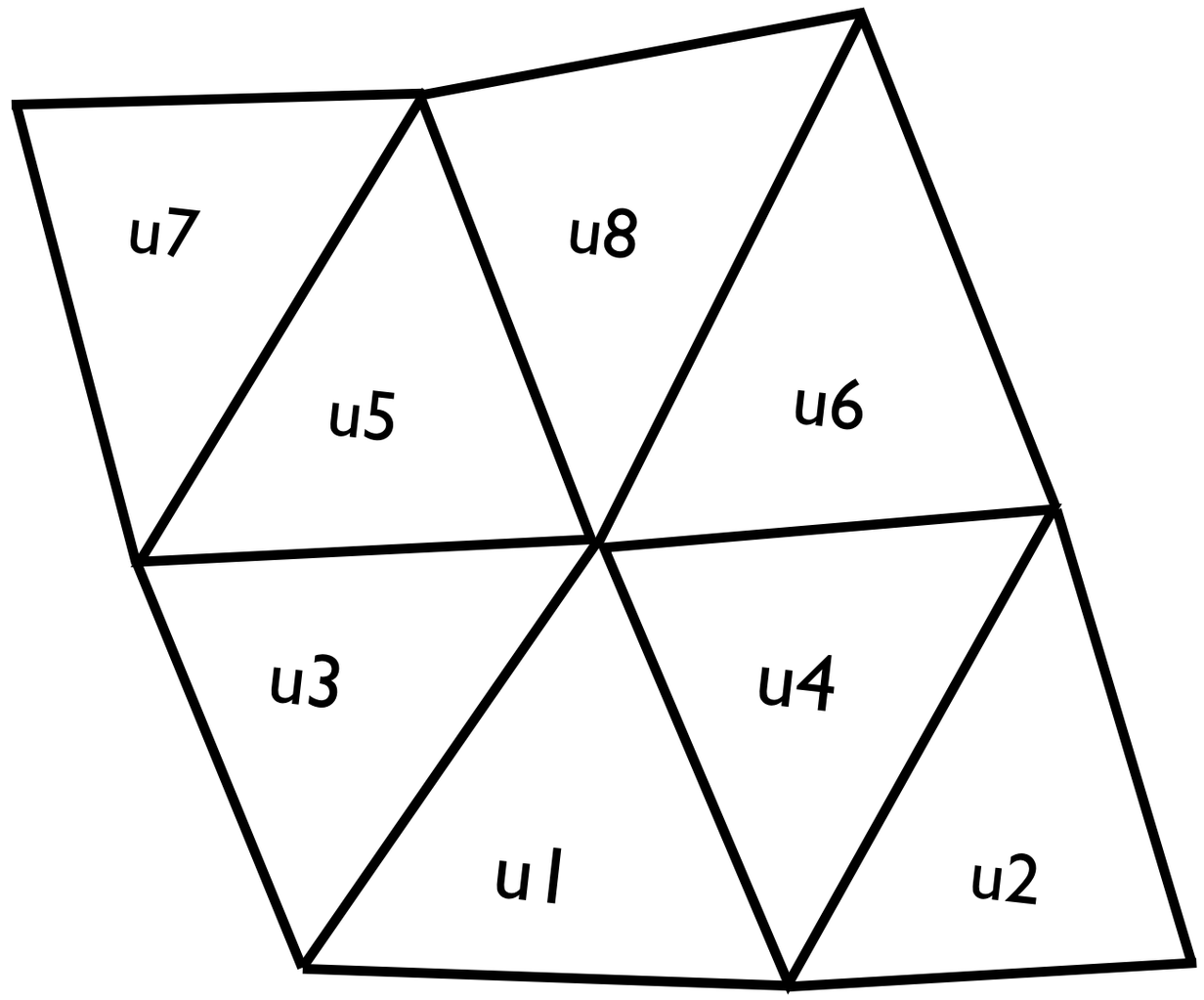
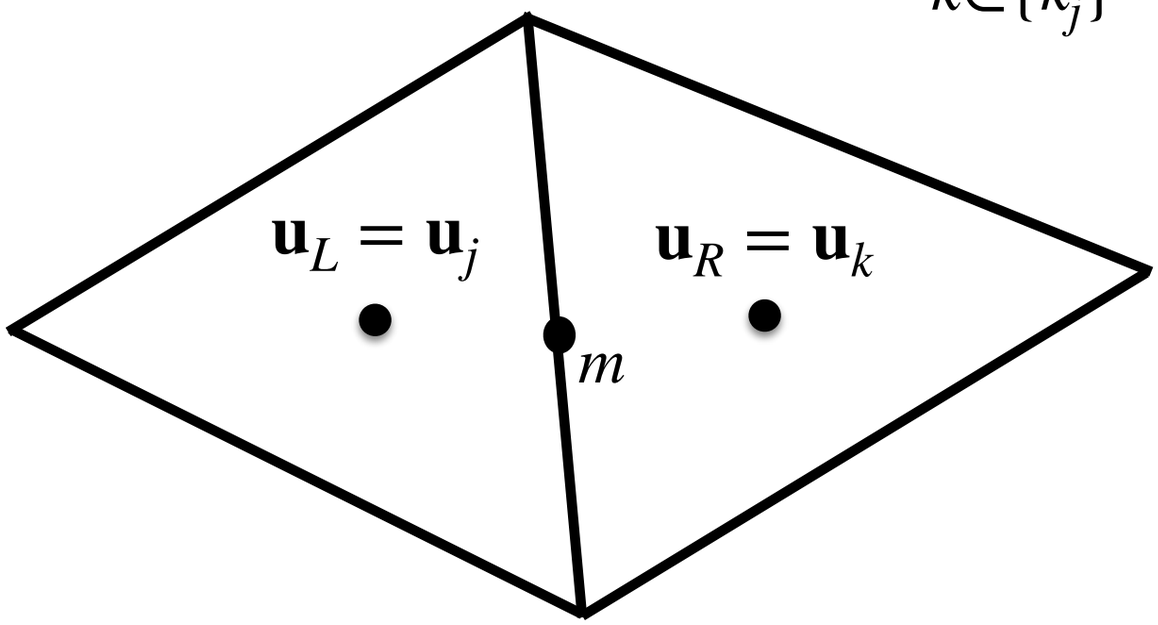
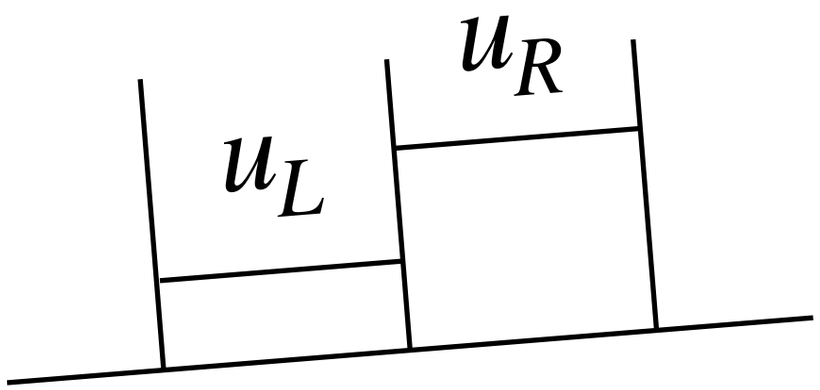
$$\partial_t \mathbf{u} + \partial_x(\mathbf{f}^{inv} + \mathbf{f}^{vis}) + \partial_y(\mathbf{g}^{inv} + \mathbf{g}^{vis}) = 0$$



Inviscid Discretization

$$\mathbf{Res}_j = \sum_{k \in \{k_j\}} \mathbf{\Phi}_n(\mathbf{u}_L, \mathbf{u}_R) dS$$

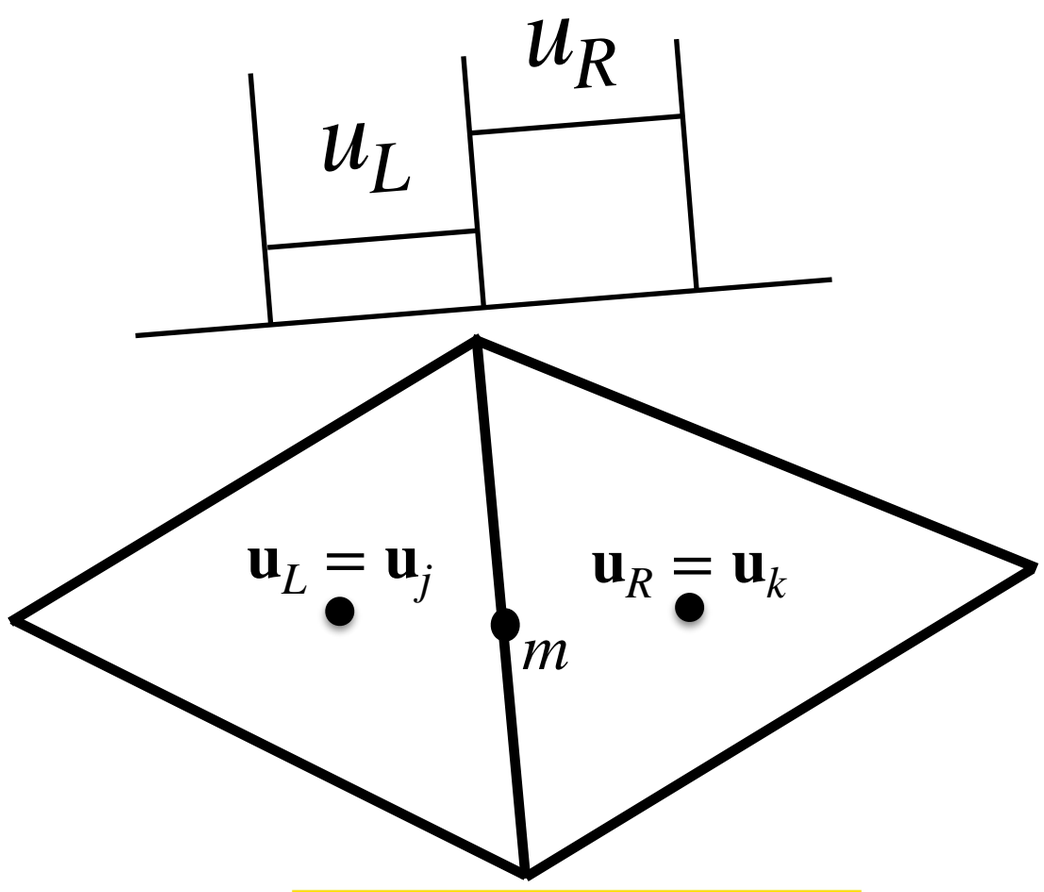


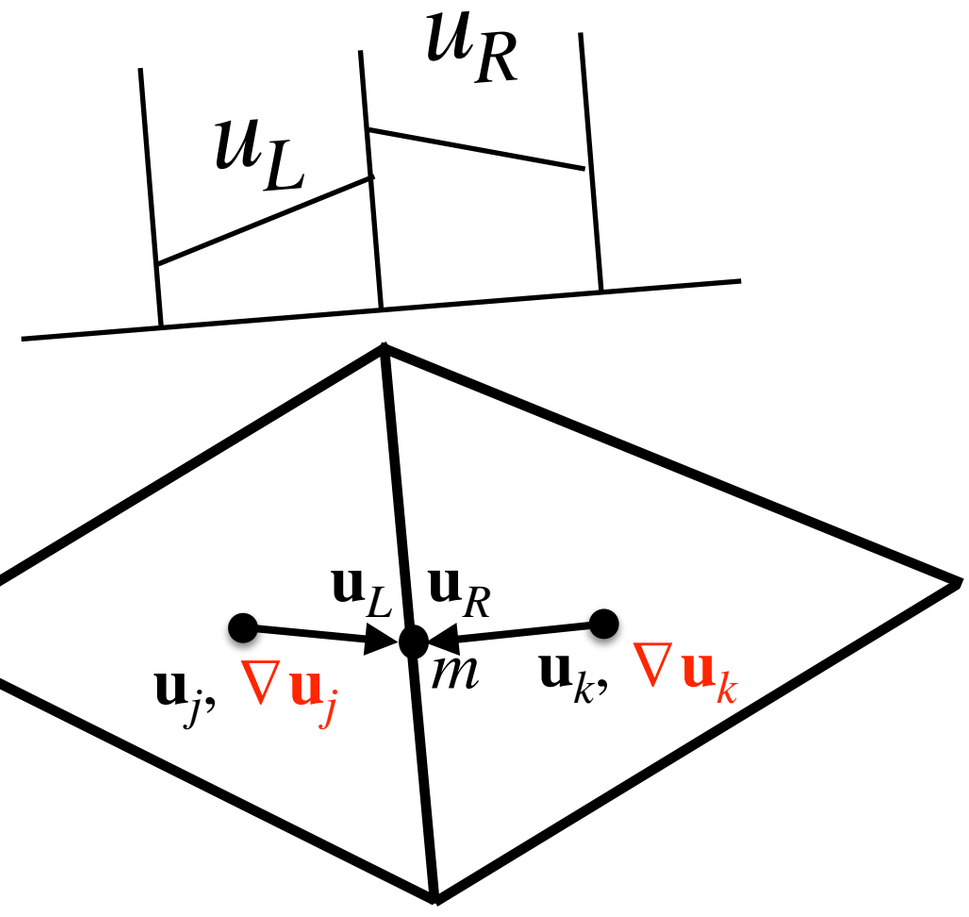


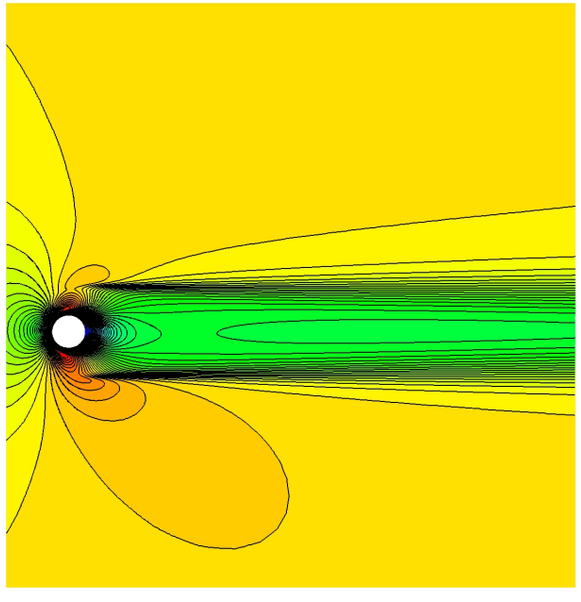
Roe: $\mathbf{\Phi}^{inv} = \frac{1}{2}\left[\mathbf{f}_R^{inv} + \mathbf{f}_L^{inv}\right] - \frac{1}{2}|\hat{\mathbf{A}}|\left(\mathbf{u}_R - \mathbf{u}_L\right)$

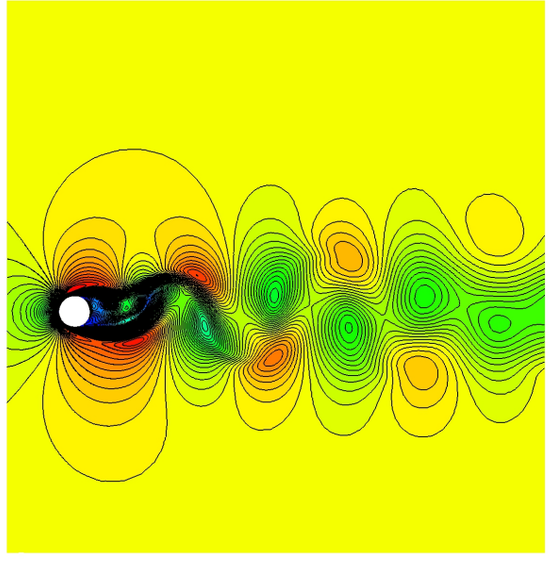
HLL: $\mathbf{\Phi}^{inv} = \frac{S_R \mathbf{f}_L^{inv} - S_L \mathbf{f}_R^{inv}}{S_R - S_L} - \frac{S_R S_L}{S_R - S_L}(\mathbf{u}_R - \mathbf{u}_L)$

2nd-order accuracy



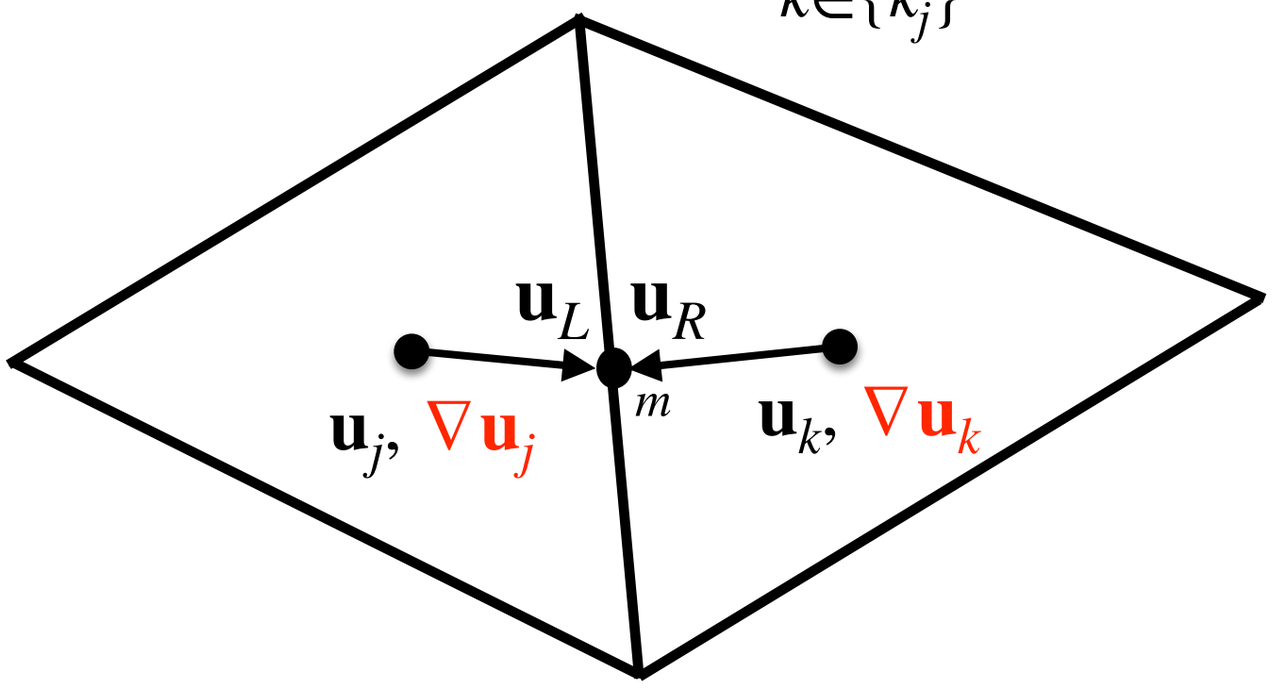


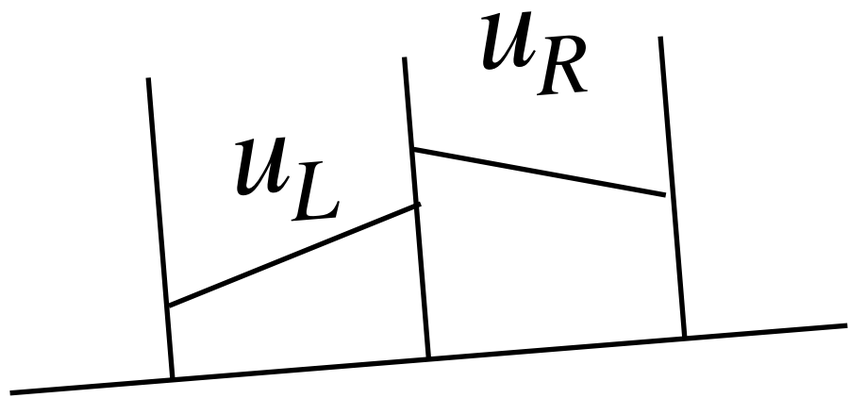




Viscous discretization:

$$\mathbf{Res}_j = \sum_{k \in \{k_j\}} \mathbf{\Phi}_n(\mathbf{u}_L, \mathbf{u}_R, \nabla \mathbf{u}_j, \nabla \mathbf{u}_k) dS$$



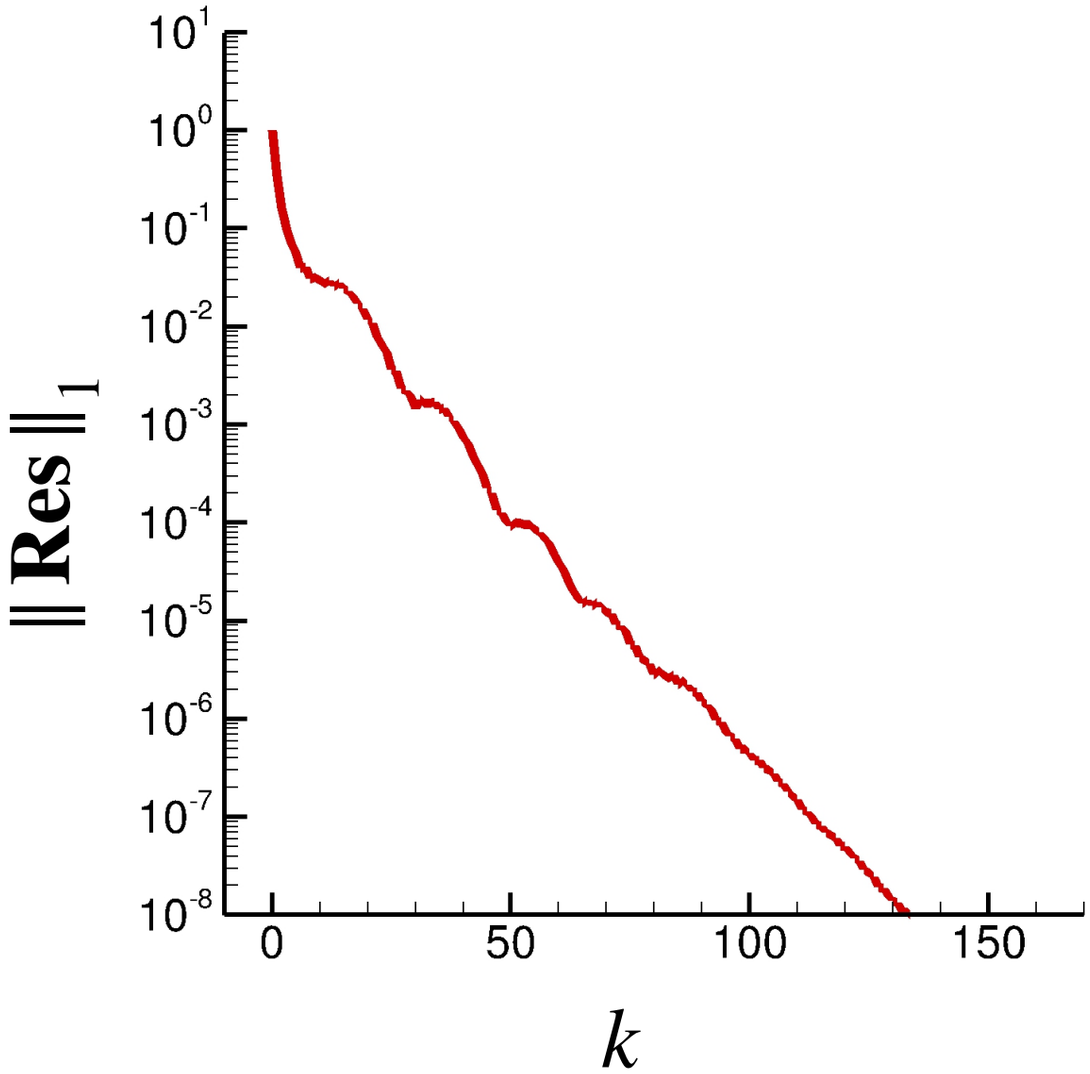


$$\mathbf{\Phi}^{vis} = \mathbf{f}^{vis}(\bar{\mathbf{u}}, \nabla \mathbf{u})$$

Nishikawa, AIAA2010-0593

$$\nabla \mathbf{u} = \frac{1}{2}\left[\nabla \mathbf{u}_j + \nabla \mathbf{u}_k\right] + \frac{(4/3)}{2\left|(\mathbf{x}_k - \mathbf{x}_j) \cdot \hat{\mathbf{n}}_{jk}\right|}\left(\mathbf{u}_R - \mathbf{u}_L\right)$$

Implicit defect-defect-correction solver

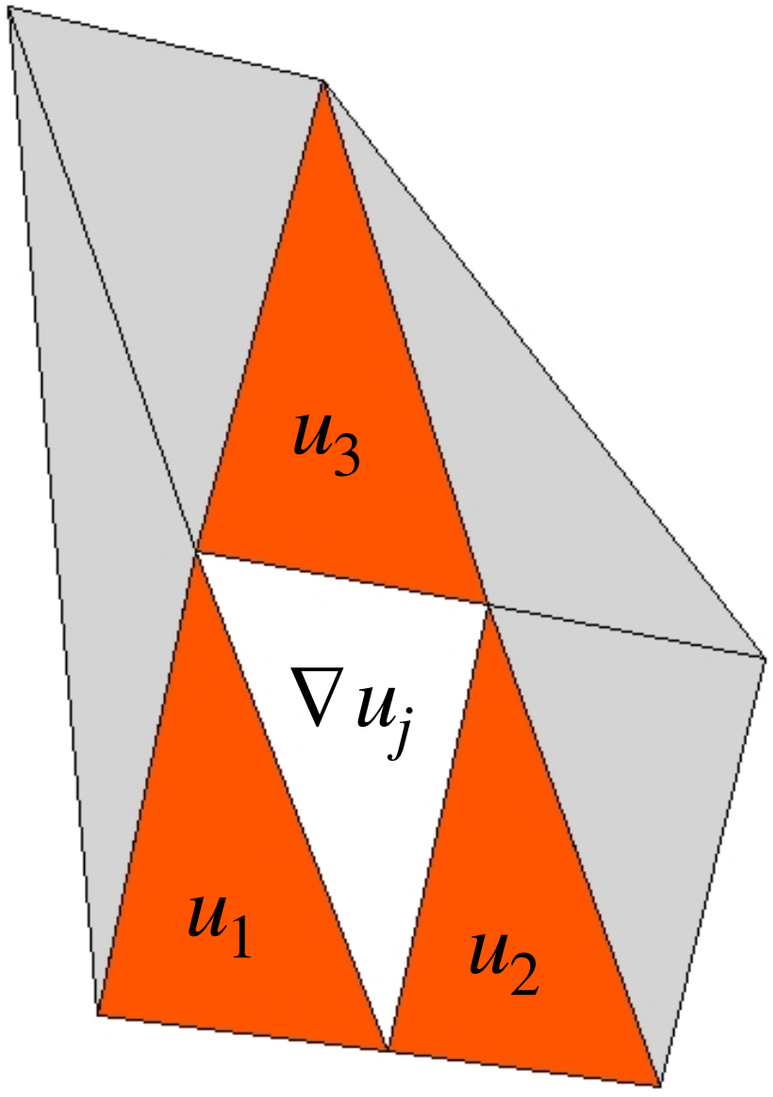


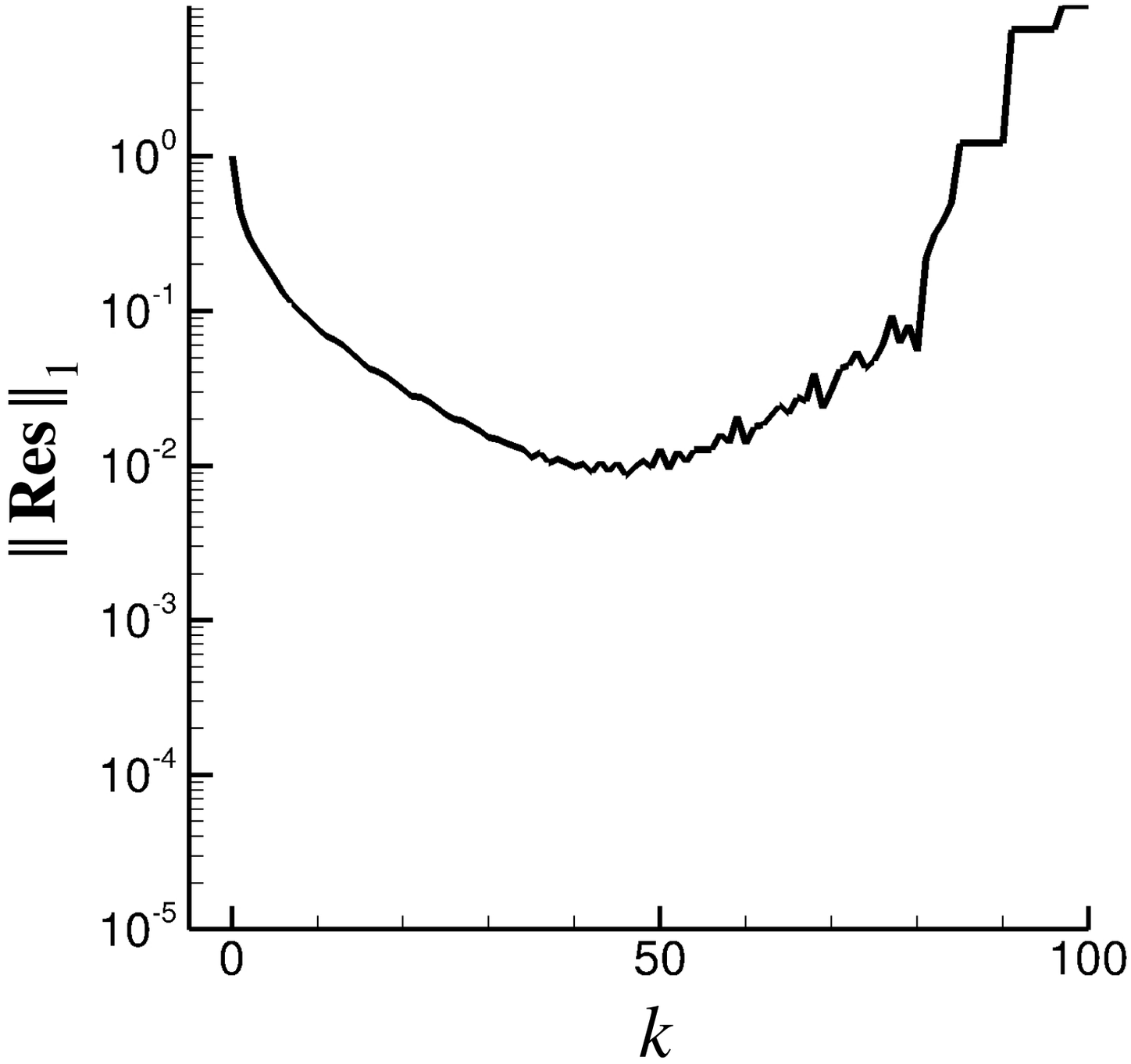
$$\mathbf{U}^{k+1} = \mathbf{U}^{k} + \Delta\mathbf{U}$$

$$\left[\frac{\mathbf{V}}{\Delta\tau} + \frac{\partial \mathbf{Res}^{(1)}}{\partial \mathbf{U}}\right] \Delta\mathbf{U} = -\,\mathbf{Res}(\mathbf{U}^{k})$$

$$\mathbf{Res}^{(1)} = \mathbf{Res}(\nabla\mathbf{U} = 0)$$

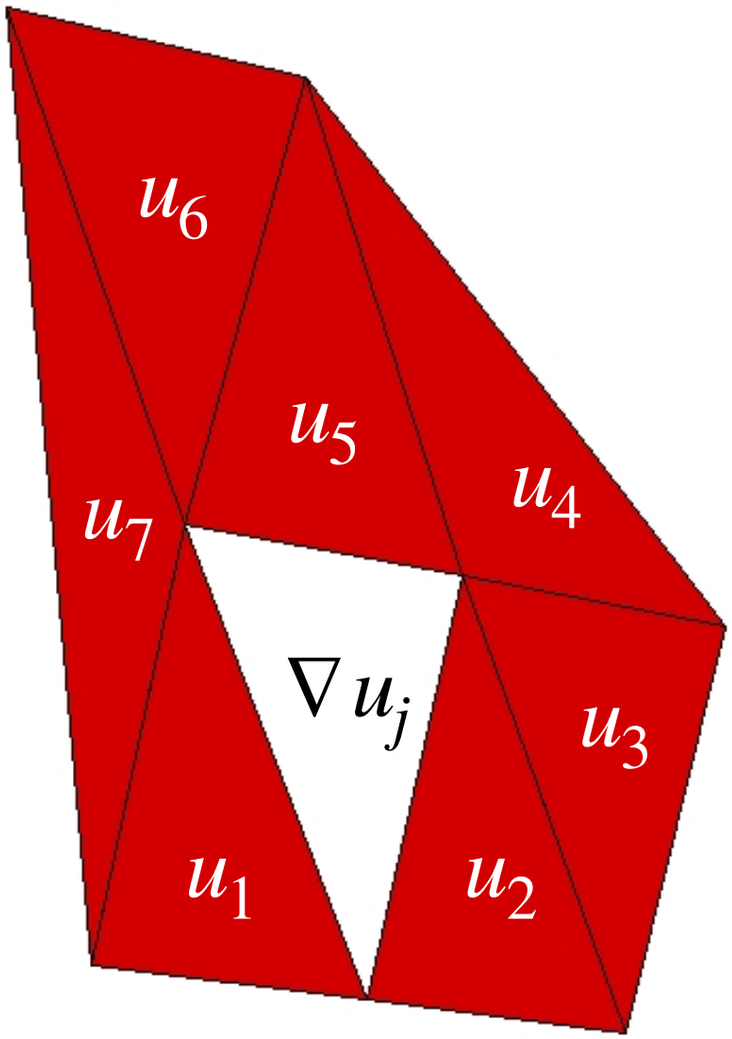
Solver gets unstable if gradients are computed with face-neighbors

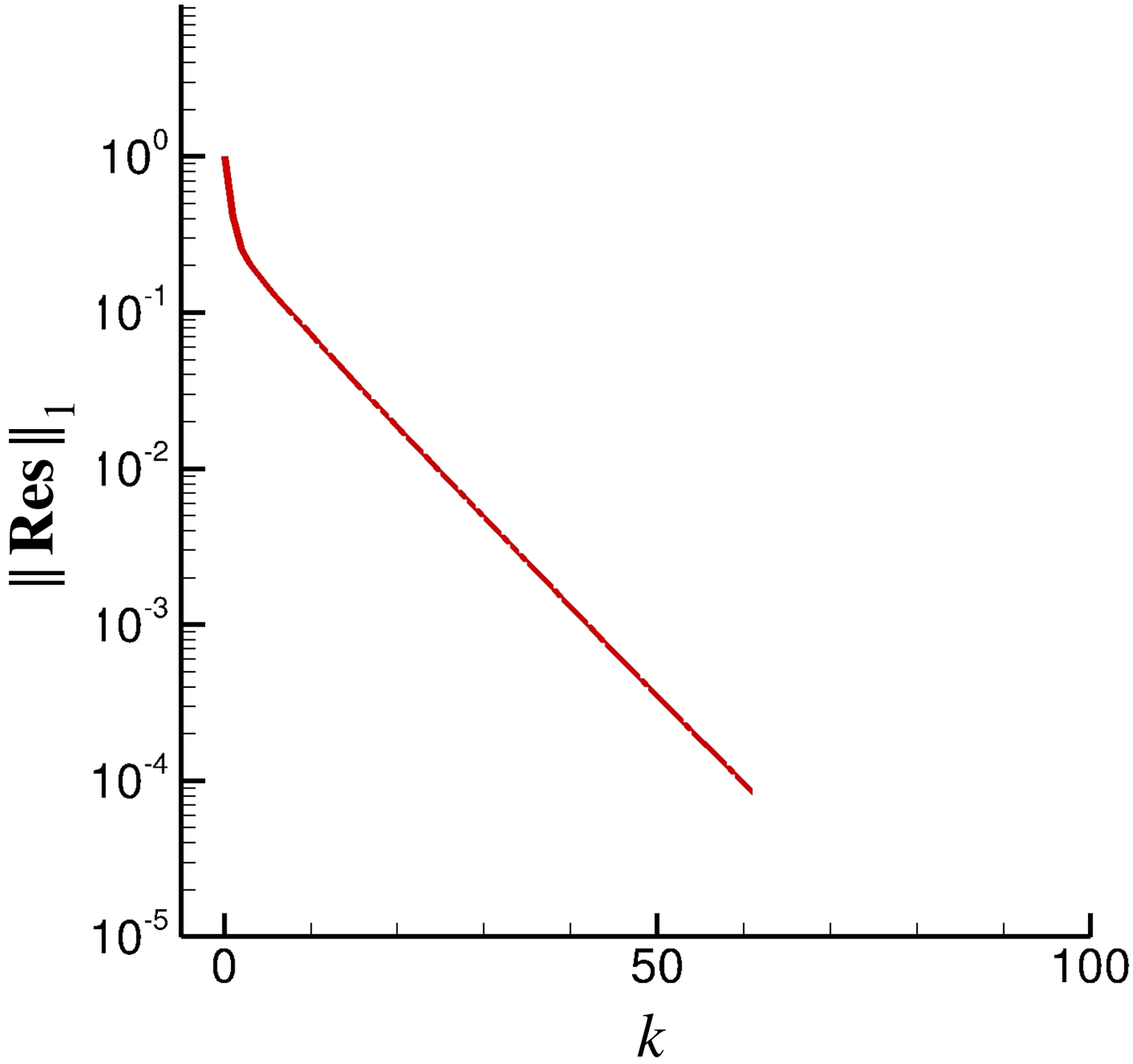




Haider, et al., Numer. Math., Vol. 113, No. 4, 2009
Diskin&Thomas, AIAA J., Vol. 49, No. 4, 2011

More neighbors for stability





Cost of Gradients

$$\begin{bmatrix} \partial_x u_j \\ \partial_y u_j \end{bmatrix} = \sum_k \begin{bmatrix} c_{jk}^x \\ c_{jk}^y \end{bmatrix} (u_k - u_j)$$

Memory

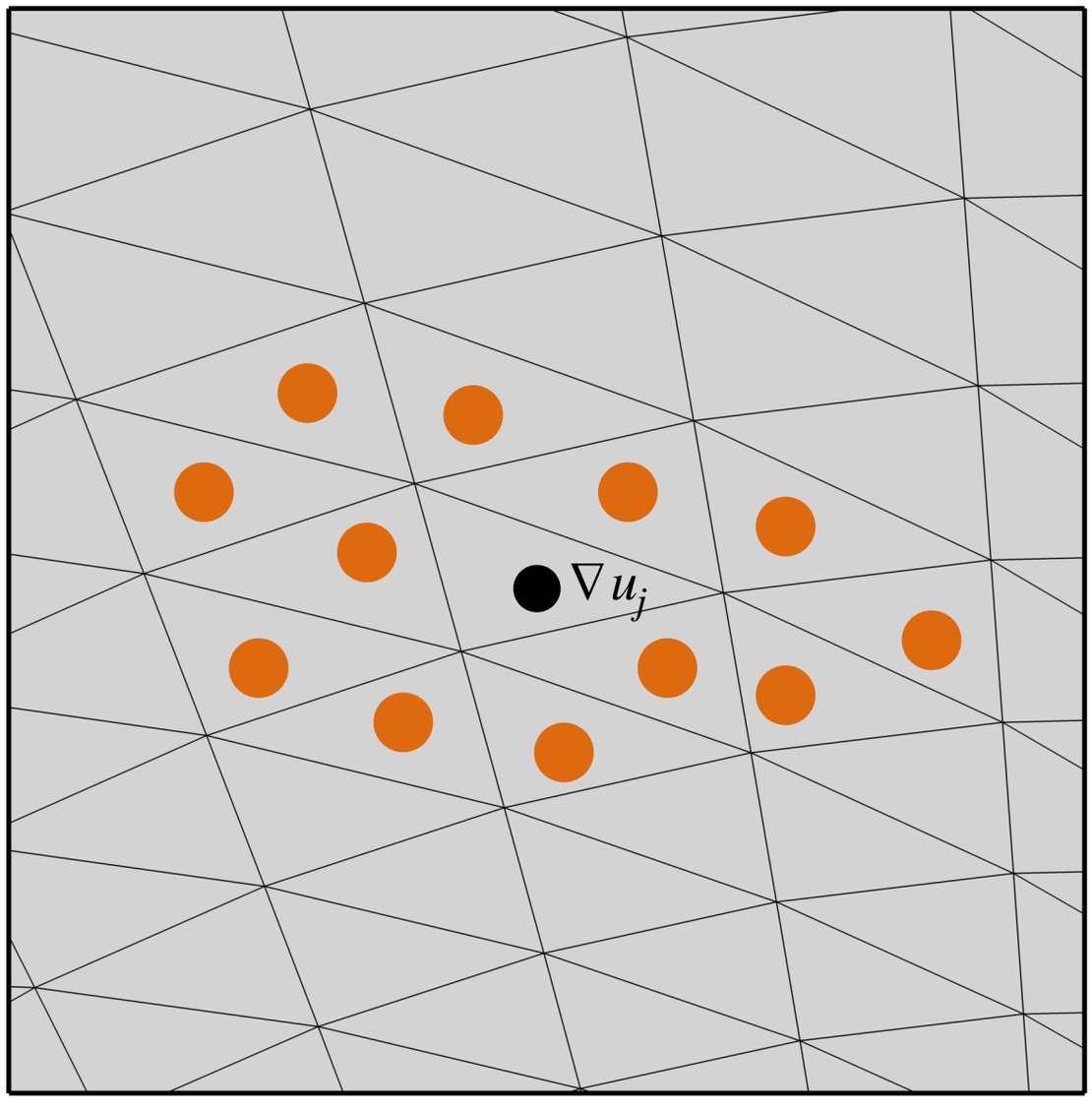
$$\begin{bmatrix} c_{jk}^x \\ c_{jk}^y \end{bmatrix} = ncells \times nnghbrs$$

CPU

```
do j = 1, ncells
 do k = 1, nnghbrs(j)
```

$$\begin{bmatrix} \partial_x u_j \\ \partial_y u_j \end{bmatrix} = \sum_k \begin{bmatrix} c_{jk}^x \\ c_{jk}^y \end{bmatrix} (u_k - u_j)$$

```
 end do
end do
```



$$\rightarrow ncells \times nnghbrs \approx 2nnodes \times 12 = \underline{24 \times nnodes}$$

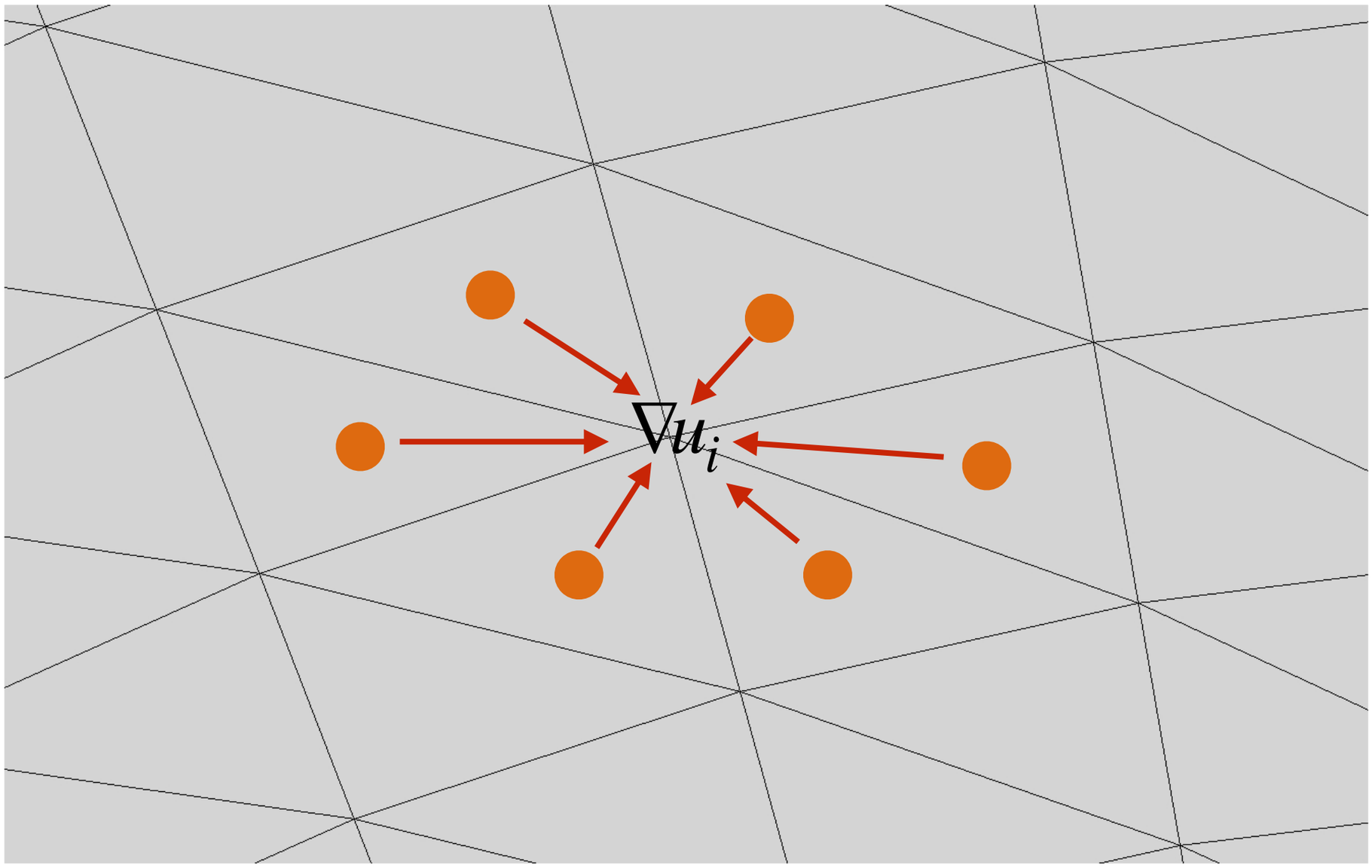
Efficient gradient method by Zhang

Zhang proposed to compute gradients at nodes

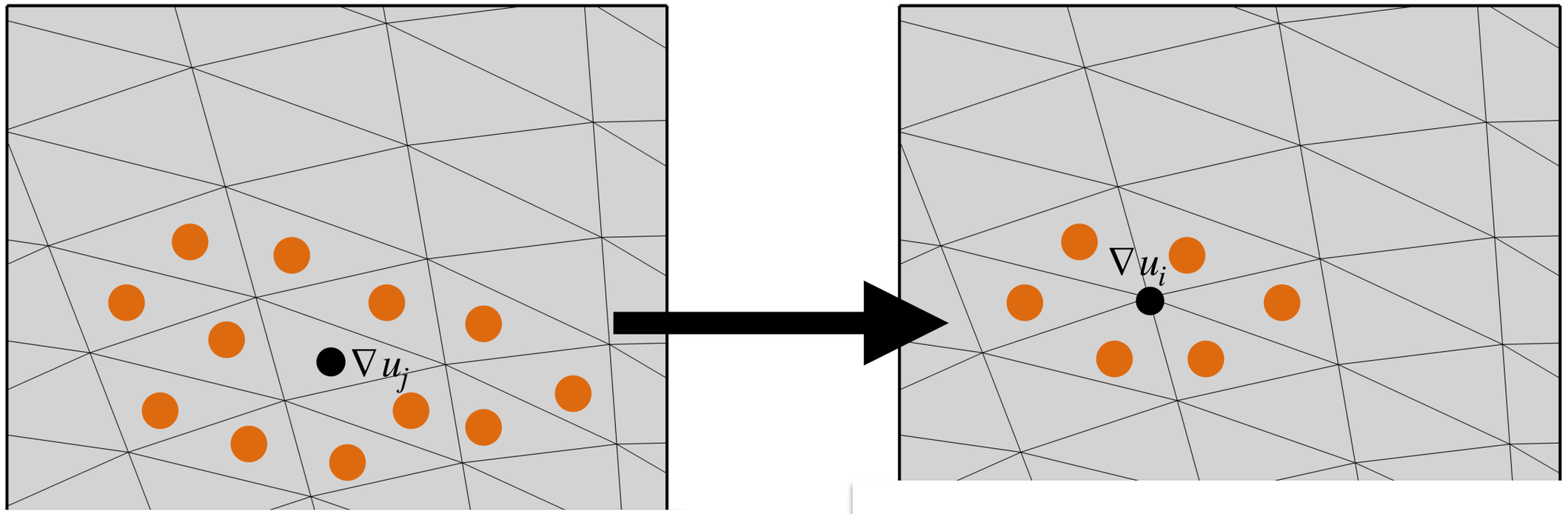
Fan Zhang (2015,2017)

"A vertex-weighted-least-squares gradient reconstruction"

https://arxiv.org/pdf/1702.04518.pdf



75% savings in cost



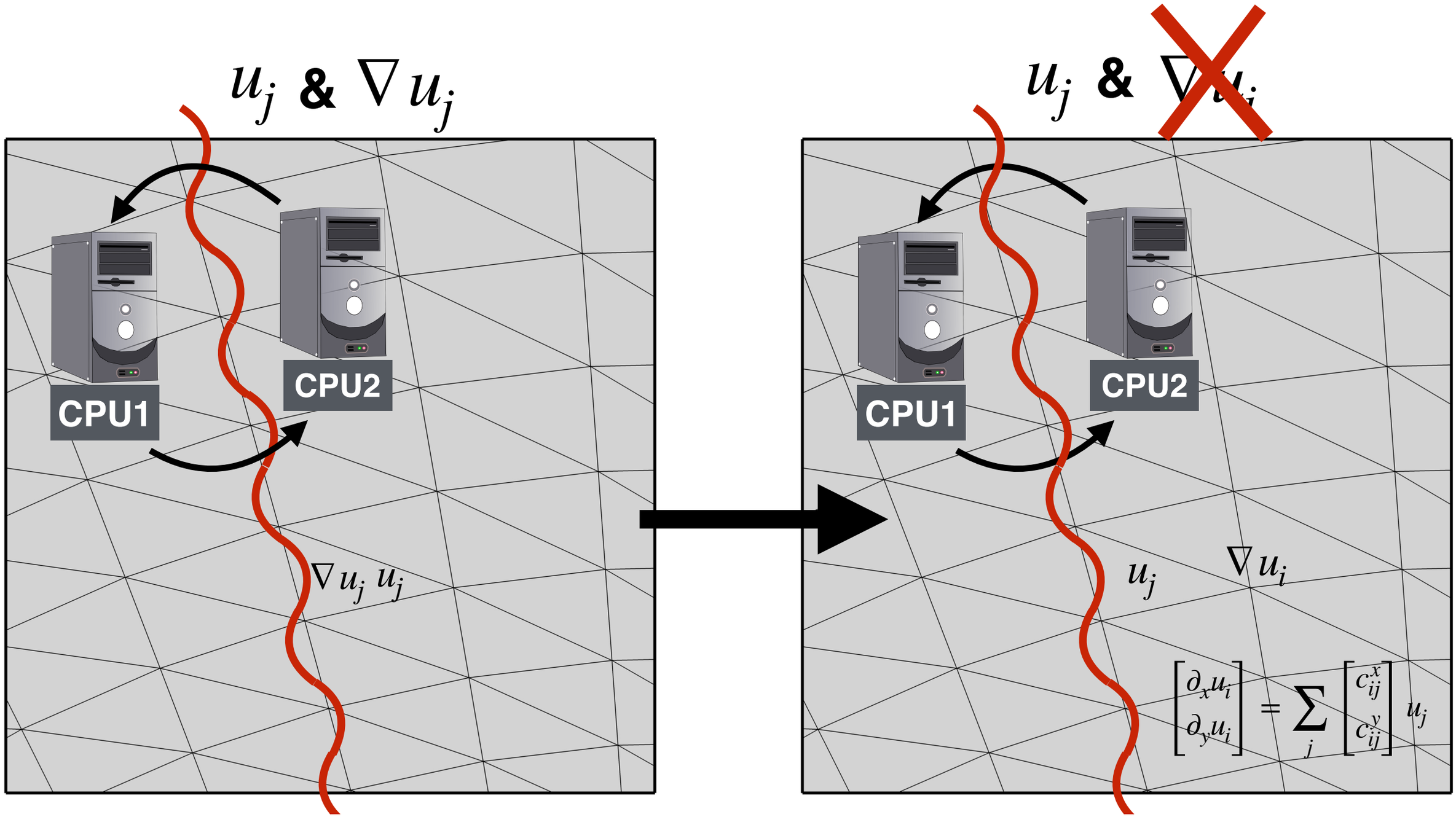
$$\begin{bmatrix} \partial_x u_j \\ \partial_y u_j \end{bmatrix} = \sum_k \begin{bmatrix} c_{jk}^x \\ c_{jk}^y \end{bmatrix} (u_k - u_j)$$

$$\begin{bmatrix} u_i \\ \partial_x u_i \\ \partial_y u_i \end{bmatrix} = \sum_j \begin{bmatrix} c_{ij}^u \\ c_{ij}^x \\ c_{ij}^y \end{bmatrix} u_j$$

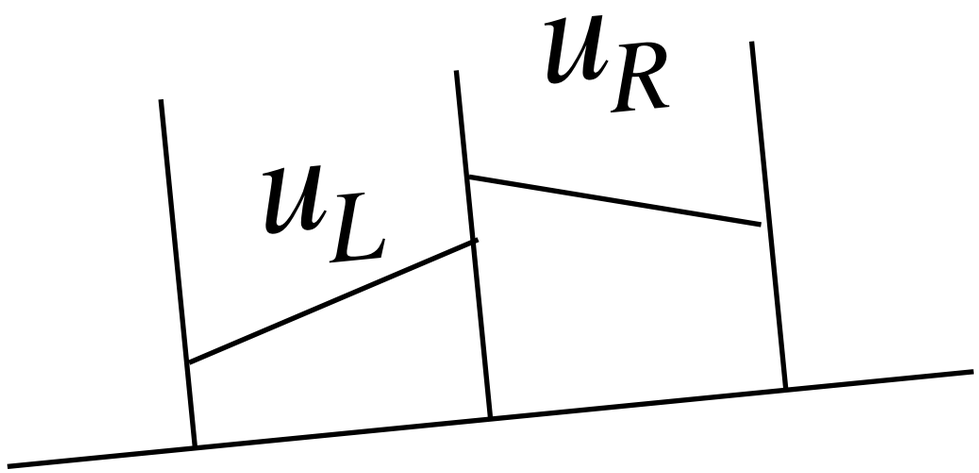
$24 \times nnodes \longrightarrow 6 \times nnodes$

$$\frac{1}{4}$$

Communication not needed

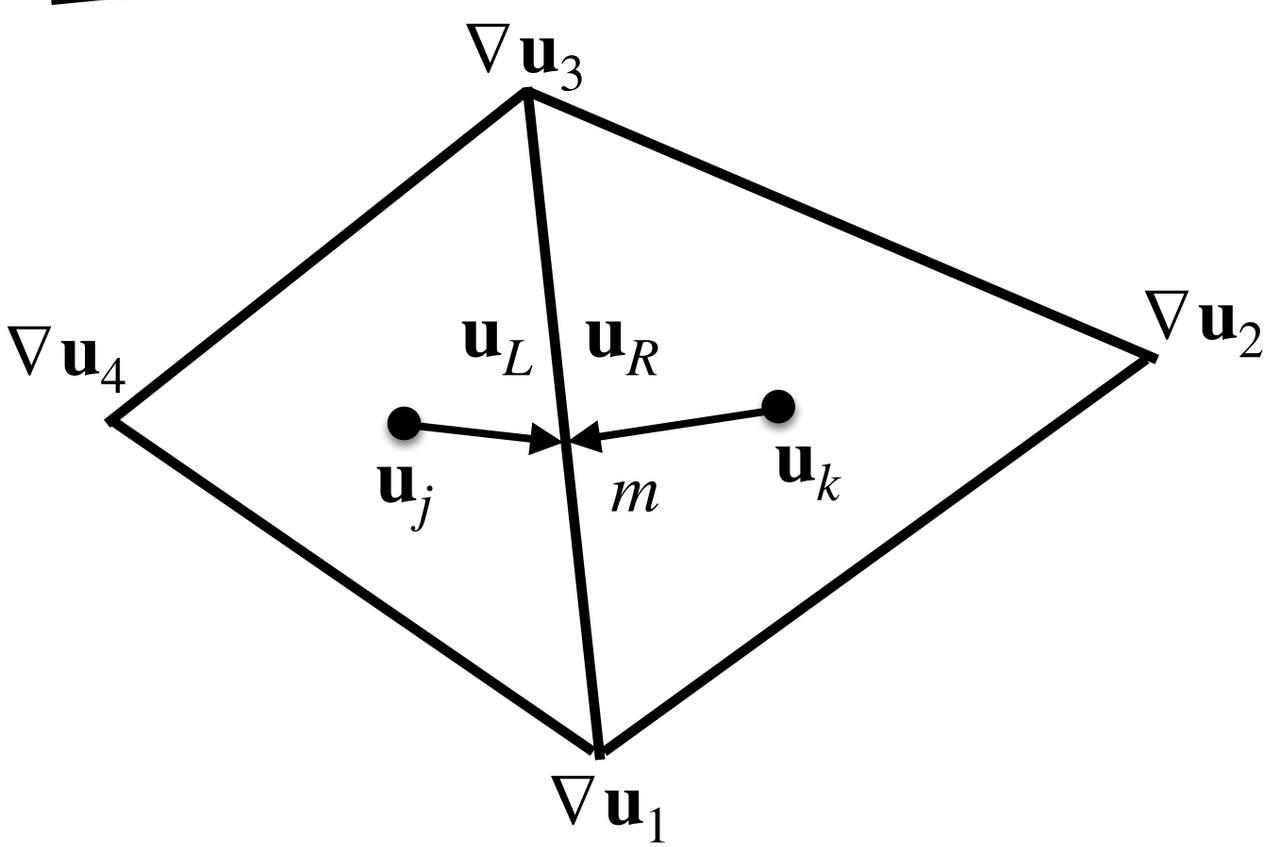


But need cell gradients…
How do we get them?



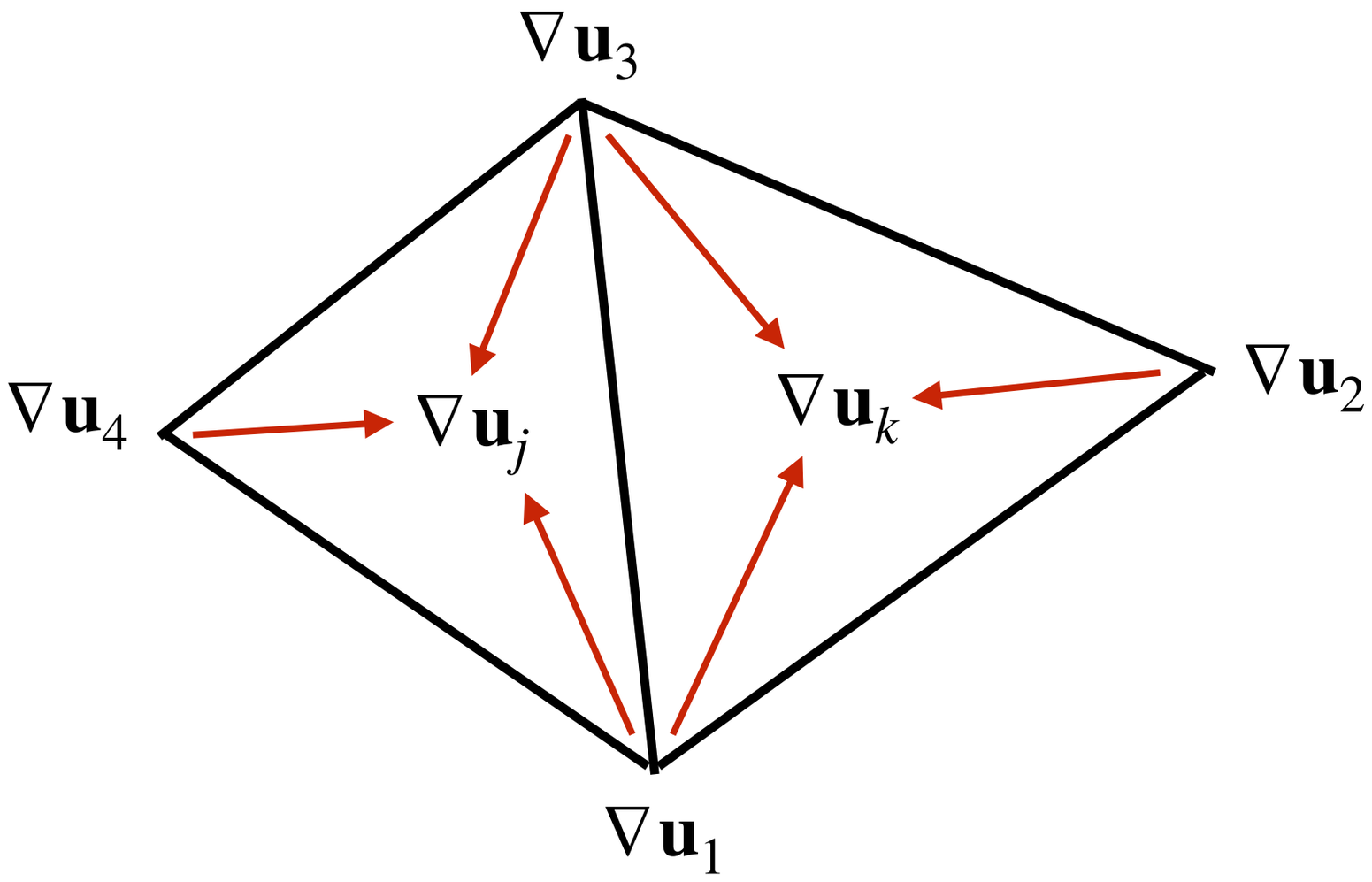
$$\mathbf{u}_L = \mathbf{u}_j + \underline{\nabla \mathbf{u}_j} \cdot (\mathbf{x}_m - \mathbf{x}_j)$$

$$\mathbf{u}_R = \mathbf{u}_k + \underline{\nabla \mathbf{u}_k} \cdot (\mathbf{x}_m - \mathbf{x}_k)$$



Fan Zhang (2015,2017)

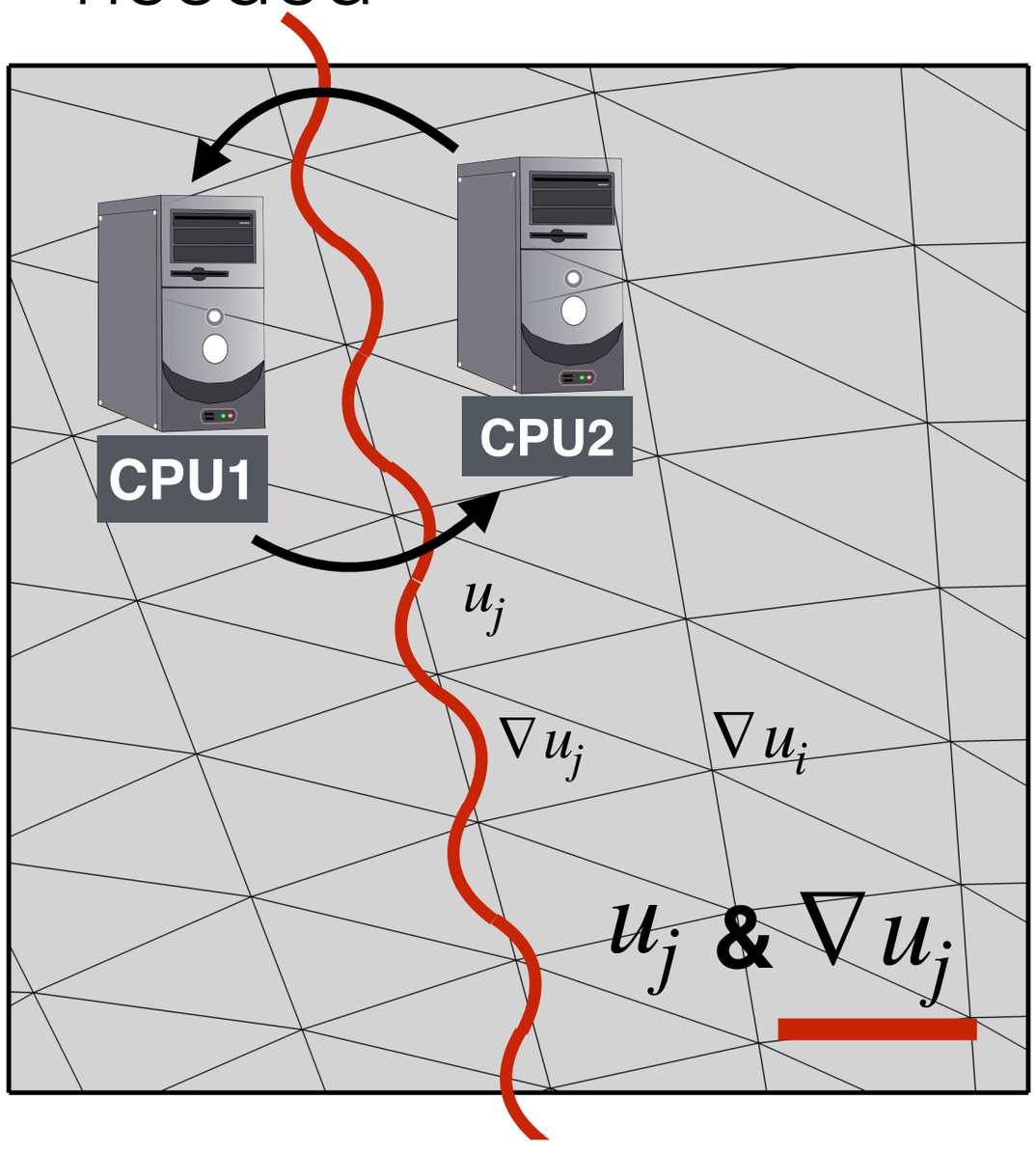
proposes the arithmetic average over each cell



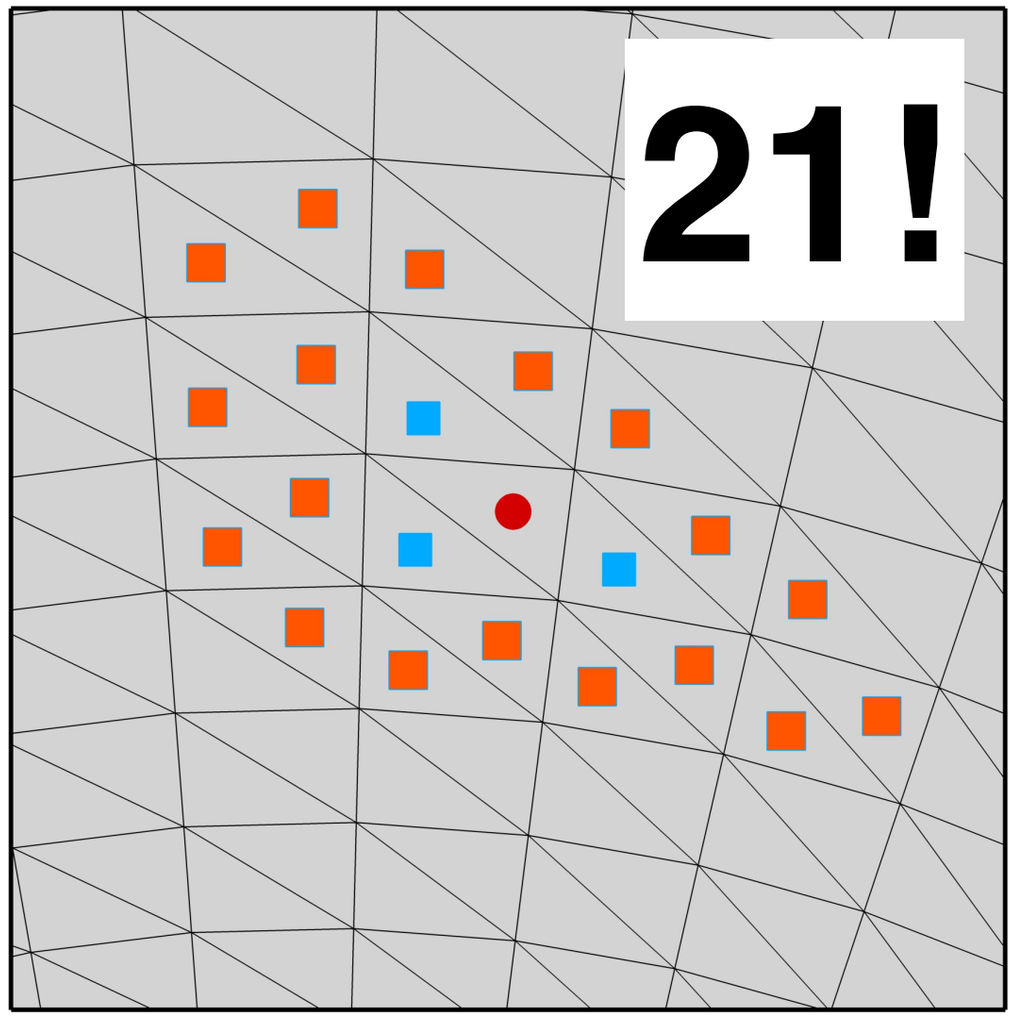
$$\nabla \mathbf{u}_j = \frac{1}{3}\left(\nabla \mathbf{u}_1 + \nabla \mathbf{u}_3 + \nabla \mathbf{u}_4\right) \qquad \nabla \mathbf{u}_k = \frac{1}{3}\left(\nabla \mathbf{u}_1 + \nabla \mathbf{u}_2 + \nabla \mathbf{u}_3\right)$$

Problems:

1. Gradient communication needed



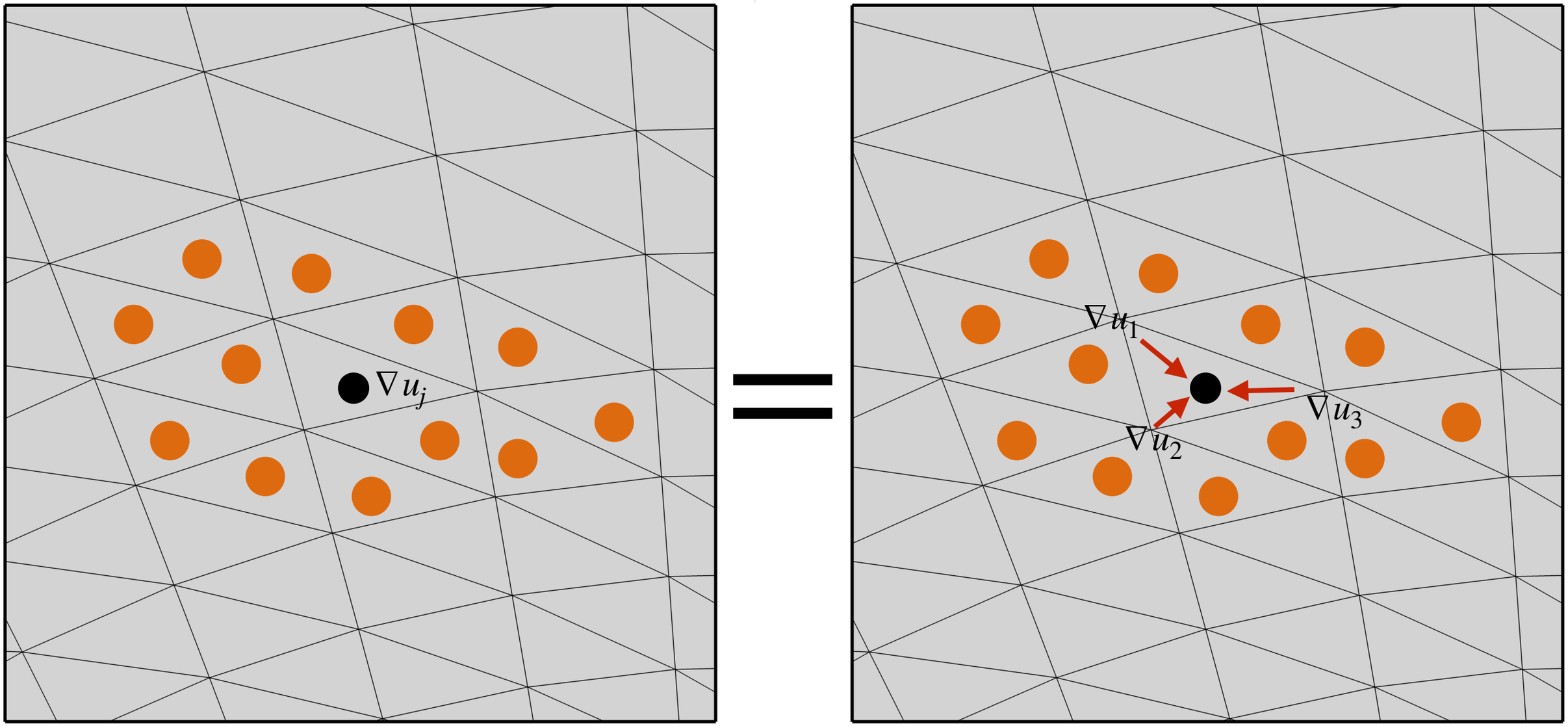
2. Large residual stencil



$$\mathbf{Res}_j = \sum_{k \in \{k_j\}} \mathbf{\Phi}_n(\mathbf{u}_L, \mathbf{u}_R) dS$$

End up with the same gradient stencil

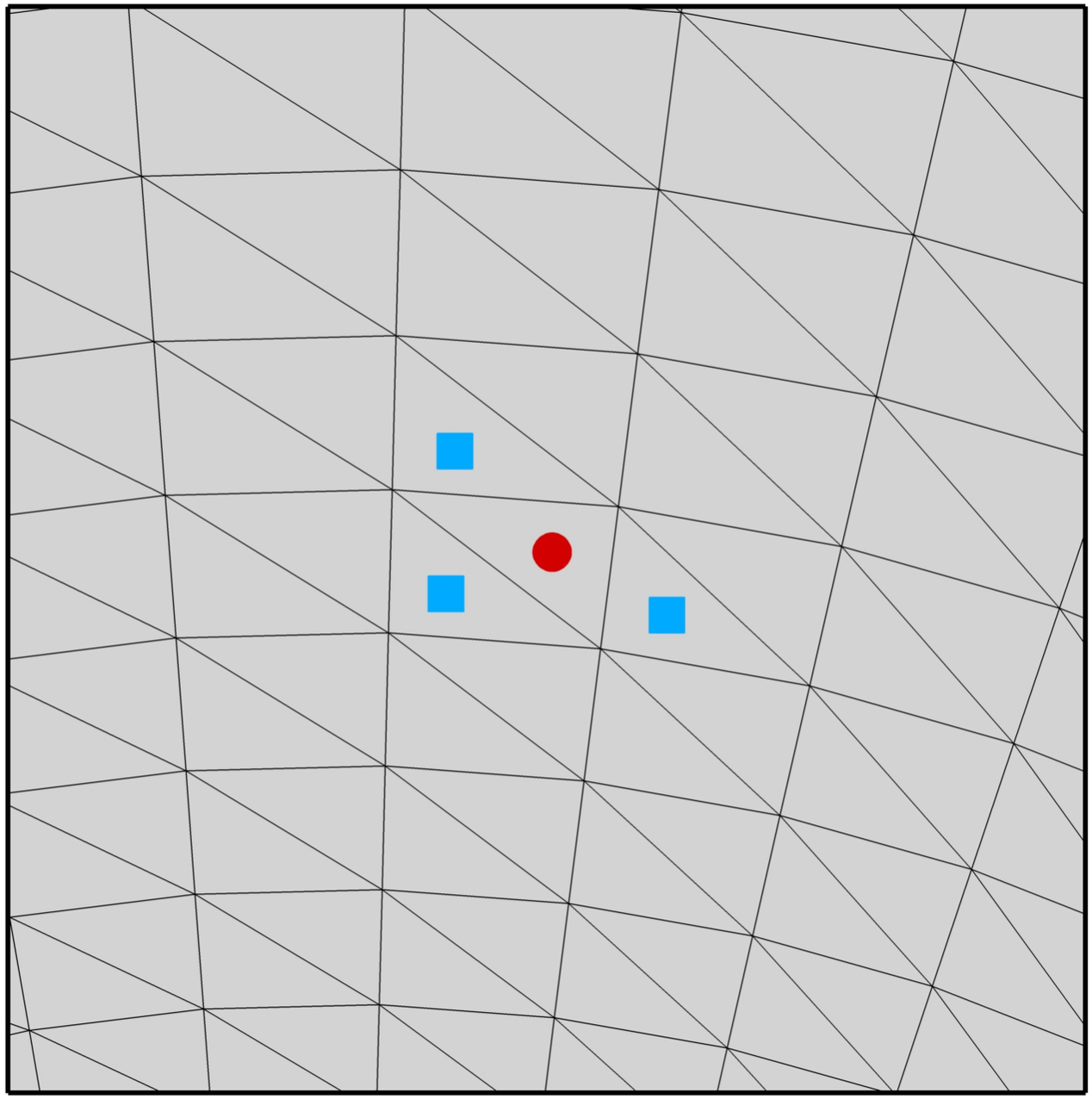
$$\mathbf{u}_L = \bar{\mathbf{u}}_j + \nabla \mathbf{u}_j \cdot (\mathbf{x}_m - \mathbf{x}_j)$$



12

12

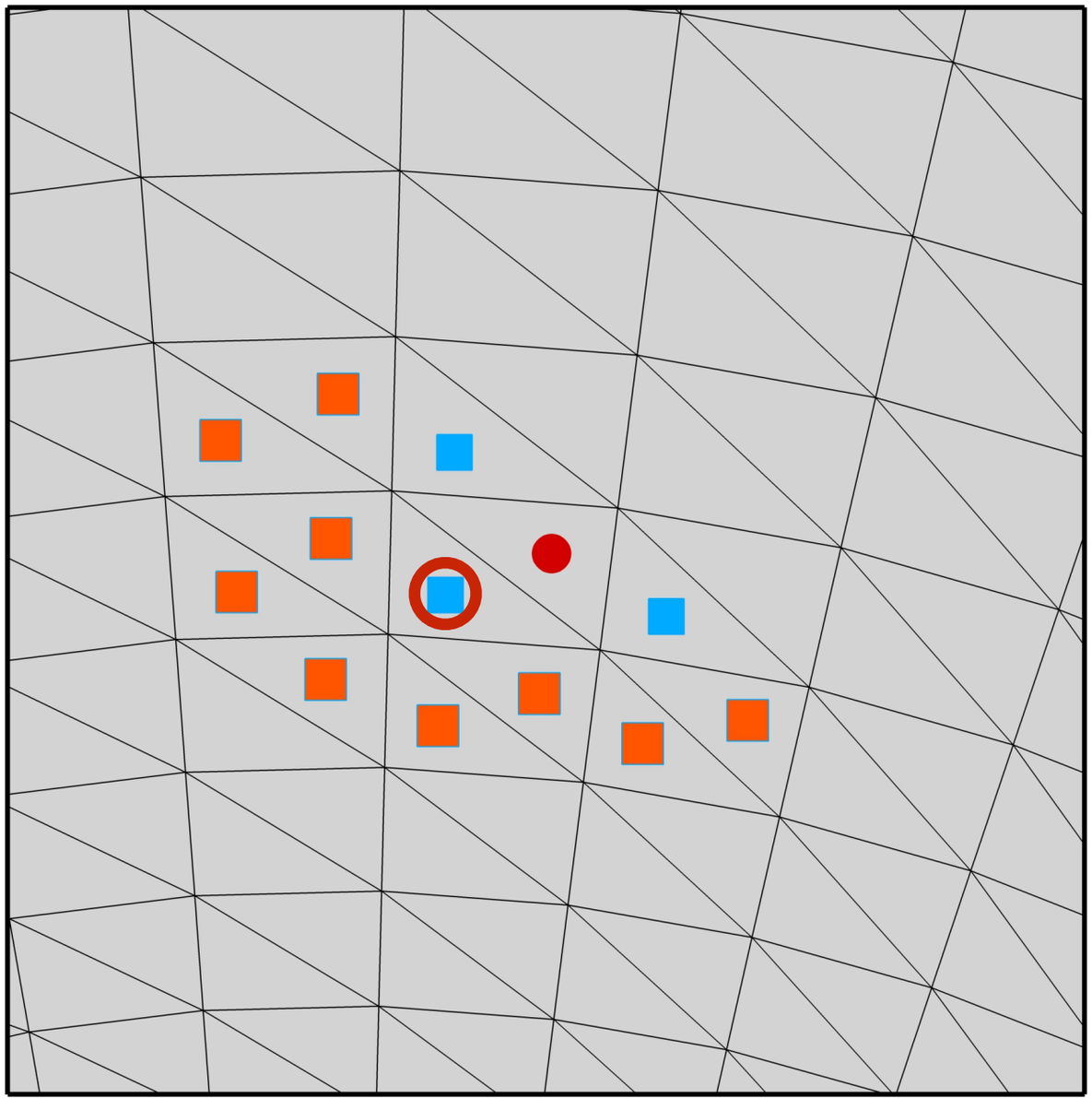
- $\mathbf{Res}_j = \sum_{k \in \{k_j\}} \mathbf{\Phi}_n(\mathbf{u}_L, \mathbf{u}_R) dS$



3

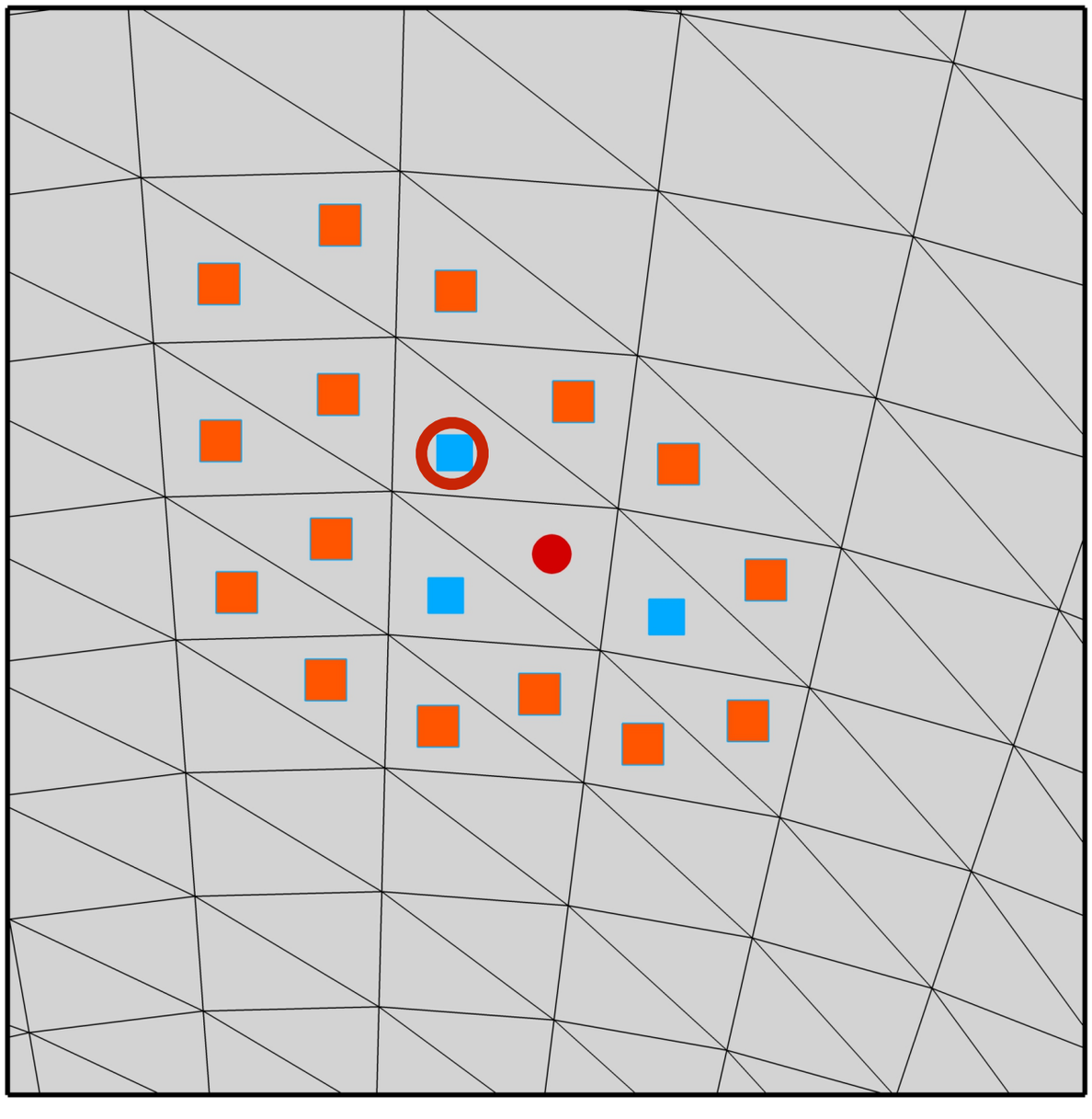
- $\mathbf{Res}_j = \sum_{k \in \{k_j\}} \mathbf{\Phi}_n(\mathbf{u}_L, \mathbf{u}_R) dS$

- $\mathbf{u}_R = \mathbf{u}_k + \nabla \mathbf{u}_k \cdot (\mathbf{x}_m - \mathbf{x}_k)$



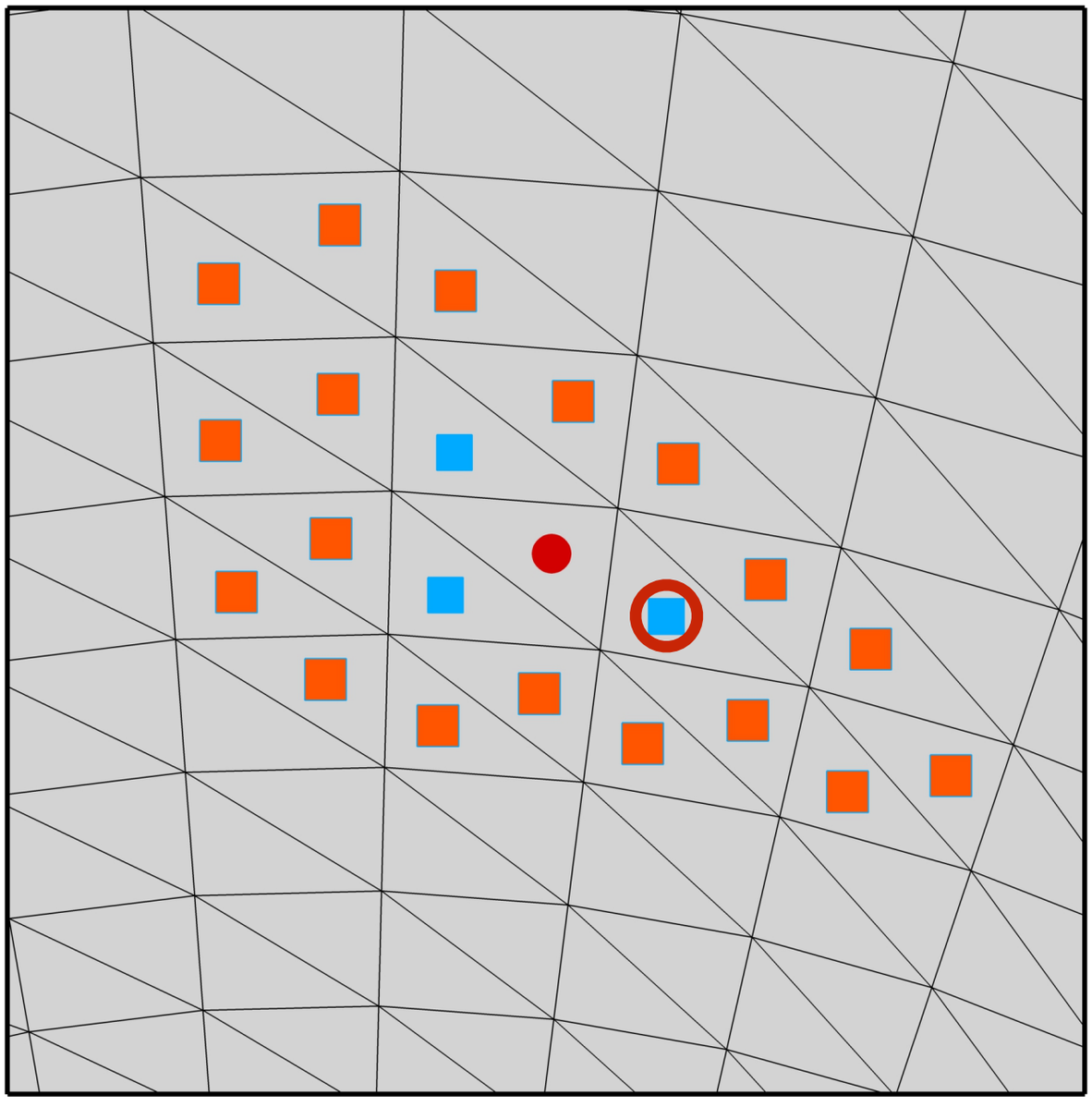
- $\mathbf{Res}_j = \sum_{k \in \{k_j\}} \mathbf{\Phi}_n(\mathbf{u}_L, \mathbf{u}_R) dS$

- $\mathbf{u}_R = \mathbf{u}_k + \nabla \mathbf{u}_k \cdot (\mathbf{x}_m - \mathbf{x}_k)$



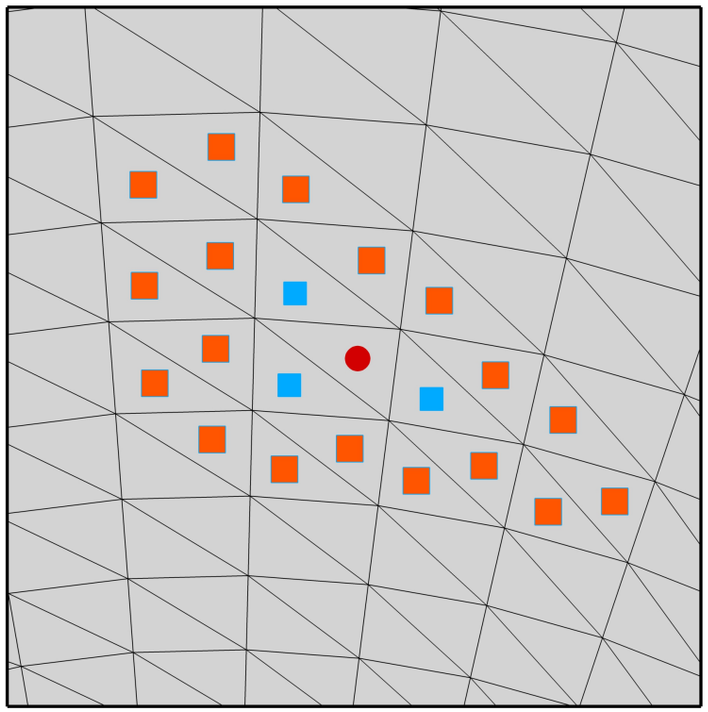
- $\mathbf{Res}_j = \sum_{k \in \{k_j\}} \Phi_n(\mathbf{u}_L, \mathbf{u}_R) dS$

- $\mathbf{u}_R = \mathbf{u}_k + \nabla \mathbf{u}_k \cdot (\mathbf{x}_m - \mathbf{x}_k)$

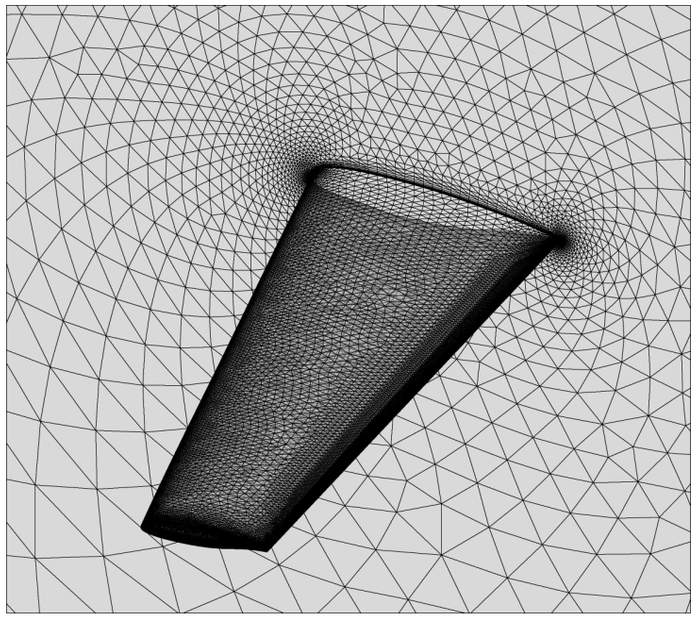


So, the residual stencil is as large as the original cell-centered method…

Large residual stencil = Large Jacobian



$$\frac{\partial \mathbf{Res}}{\partial \mathbf{U}}_{j,k} = (0 \overbrace{\underset{k=1}{\times} \quad \underset{k=2}{\times} \quad \times \quad \cdots \quad \times \quad \times}^{\mathbf{21}} \quad \cdots \quad 0)$$

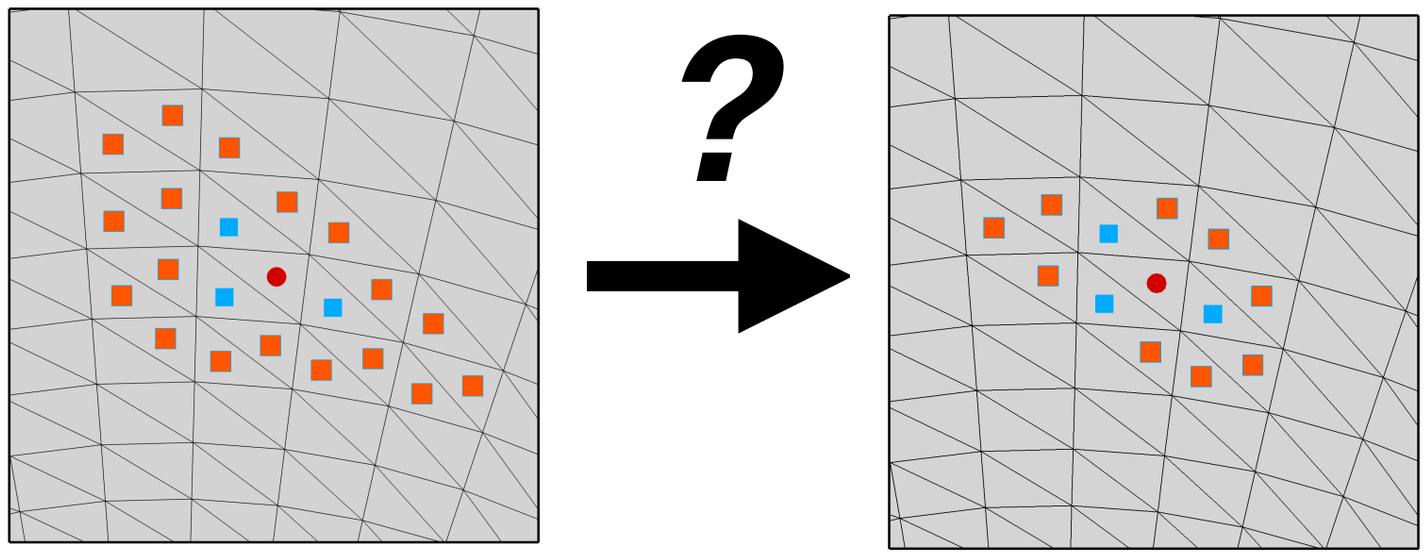


$$\frac{\partial \mathbf{Res}}{\partial \mathbf{U}}_{j,k} = (0 \overbrace{\times \quad \times \quad \times \quad \cdots \quad \times \quad \times}^{\mathbf{\sim 100}} \quad \cdots \quad 0)$$

Better to reduce it for adjoint, Newton solver, etc.

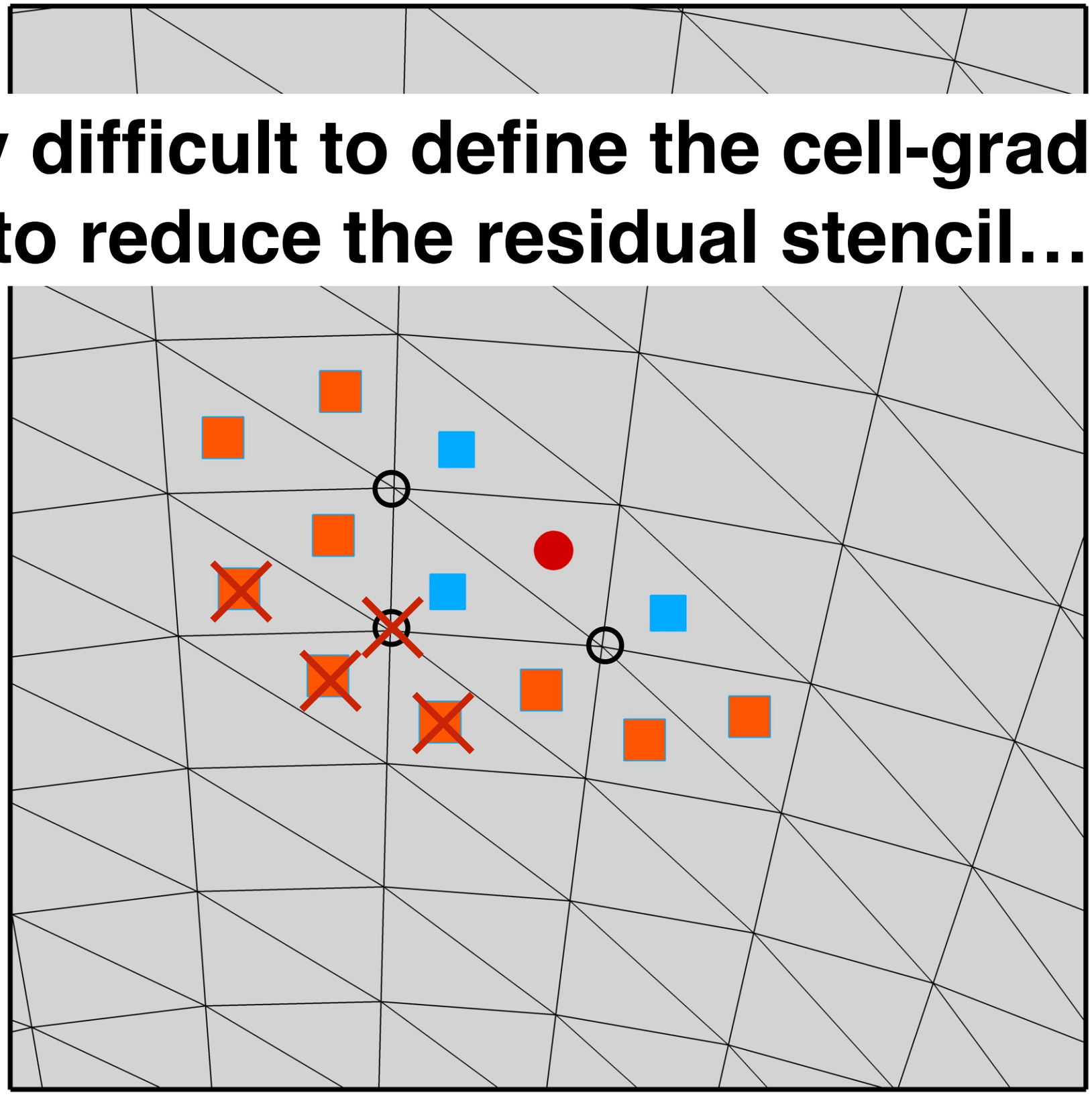
How can we reduce the residual stencil?

- $\mathbf{Res}_j = \sum_{k \in \{k_j\}} \mathbf{\Phi}_n(\mathbf{u}_L, \mathbf{u}_R) dS$



$$\frac{\partial \mathbf{Res}}{\partial \mathbf{U}} = \begin{vmatrix} x\ x\ x\ x\ x\ \ldots x\ x\ x \\ x\ x\ x\ x \ldots\ x\ x\ x \\ x\ x\ x\ \ldots\ x\ x\ x\ x \\ x\ x\ x\ x\ \ldots\ x\ x\ x \\ \ddots \\ x\ x\ x\ x\ \ldots\ x\ x\ x\ x \end{vmatrix} \quad \frac{\partial \mathbf{Res}}{\partial \mathbf{U}} = \begin{vmatrix} x\ x\ \ldots x \\ x\ \ldots x \\ x\ \ldots x\ x \\ x\ x \ldots\ x \\ \ddots \\ x\ x\ \ldots x \end{vmatrix}$$

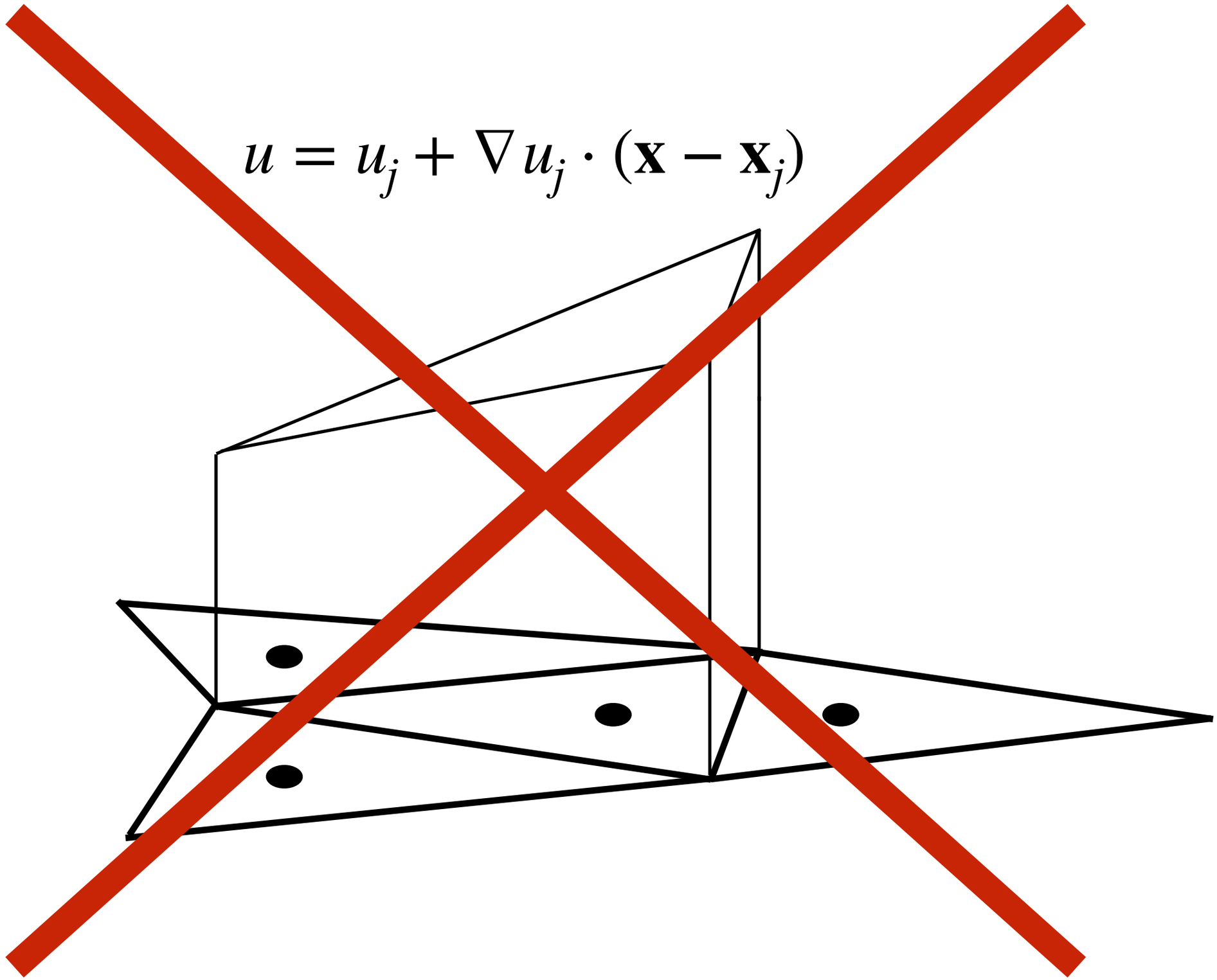
Efficient reconstruction scheme



Very difficult to define the cell-gradient to reduce the residual stencil…

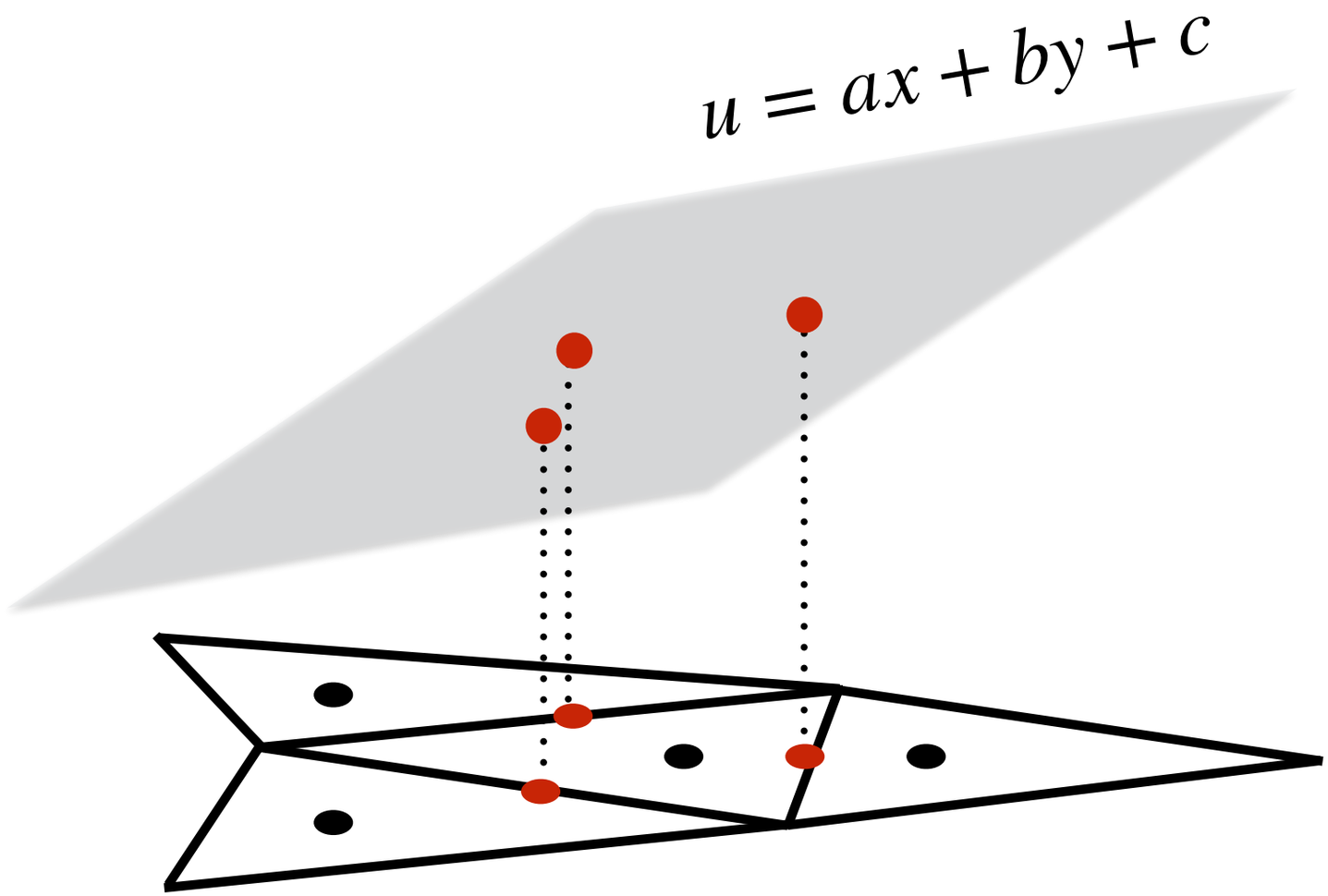
Stop thinking about cell gradients

$$u = u_j + \nabla u_j \cdot (\mathbf{x} - \mathbf{x}_j)$$

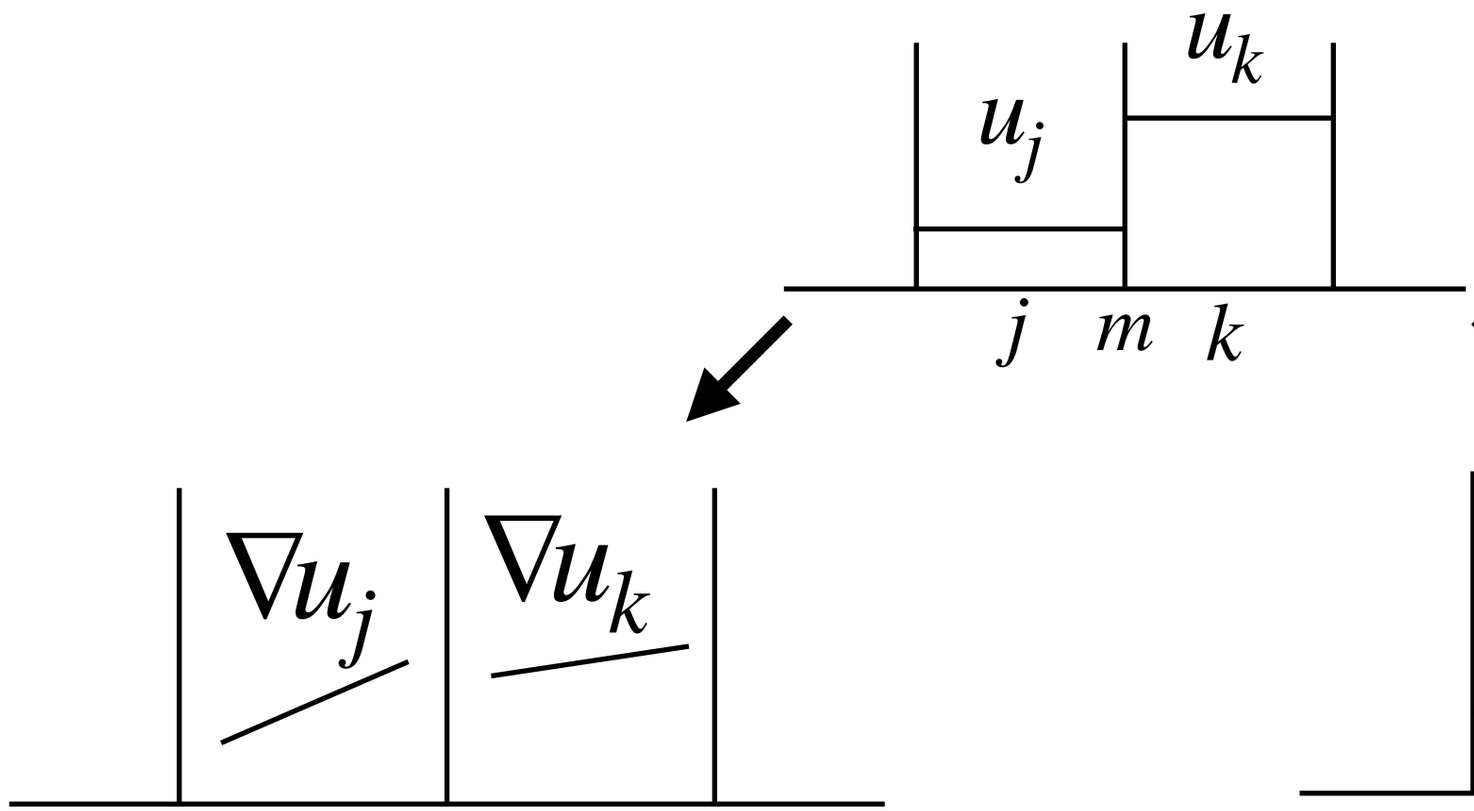


Focus on reconstructed values

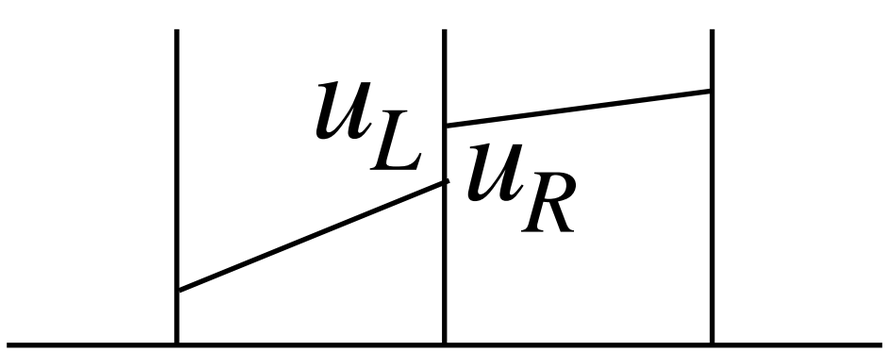
What is essential is to reconstruct the solution such that it is exact for linear functions. The linear polynomial in a cell is not essential.

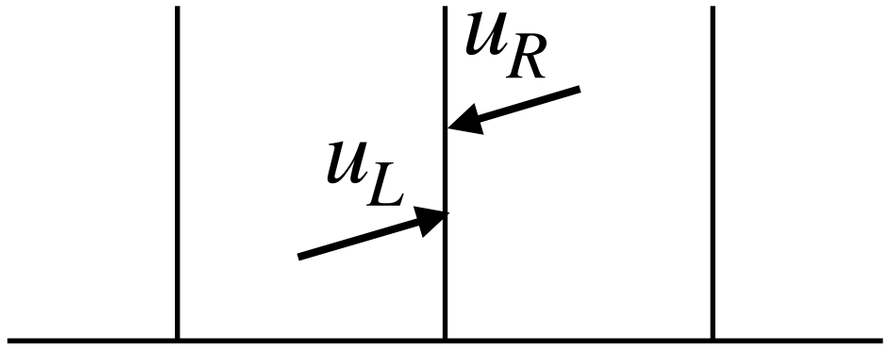


Define the gradient at the face not at the cell



This can still be exact for linear functions



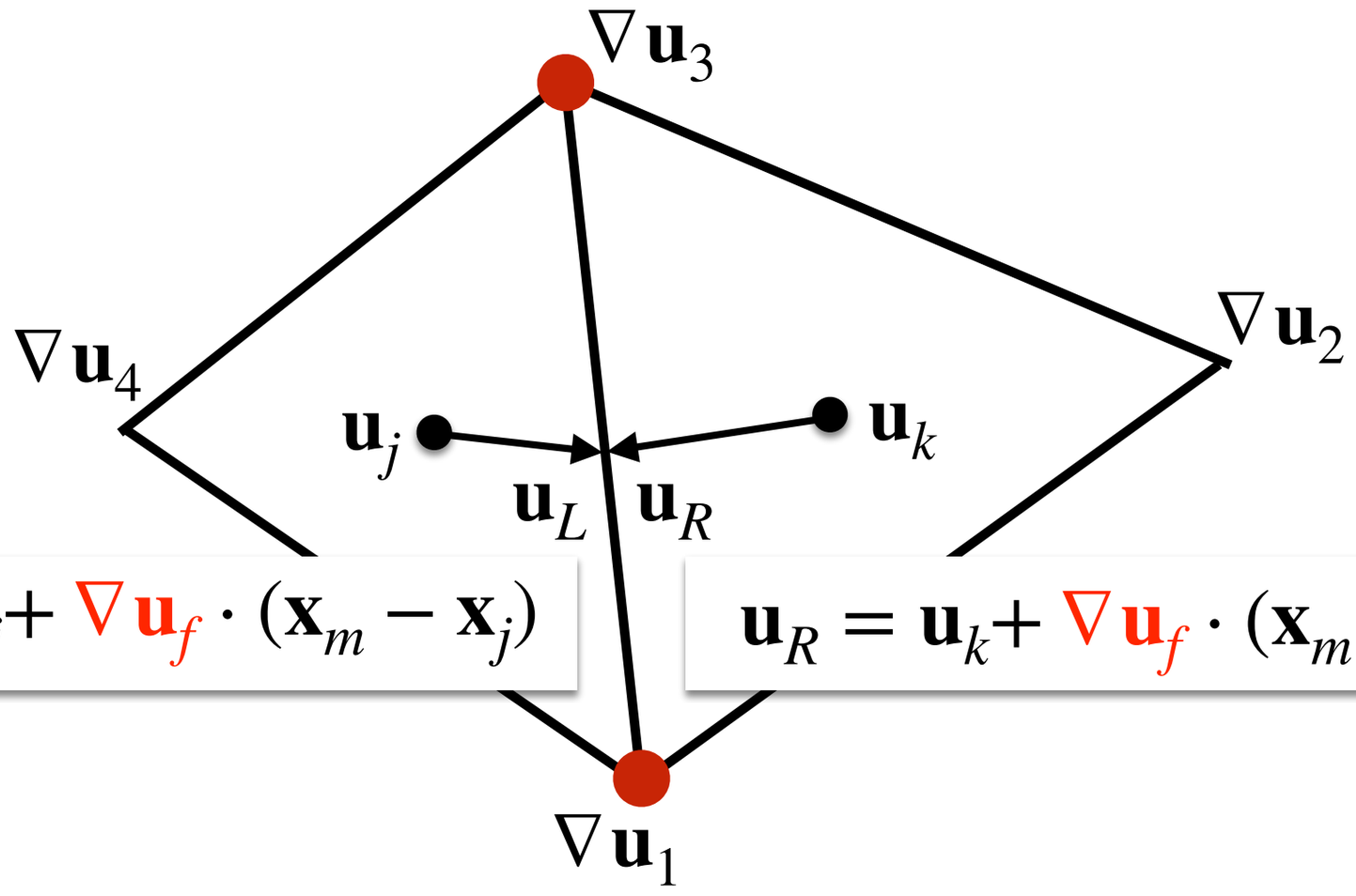


$$u_L = u_j + \nabla u_j \cdot (\mathbf{x}_m - \mathbf{x}_j)$$
$$u_R = u_k + \nabla u_k \cdot (\mathbf{x}_m - \mathbf{x}_k)$$

$$u_L = u_j + \nabla u_f \cdot (\mathbf{x}_m - \mathbf{x}_j)$$
$$u_R = u_k + \nabla u_f \cdot (\mathbf{x}_m - \mathbf{x}_k)$$

Reconstruction with common gradients

We propose to define the face gradient by averaging and use it in both reconstructions from the left and right cells

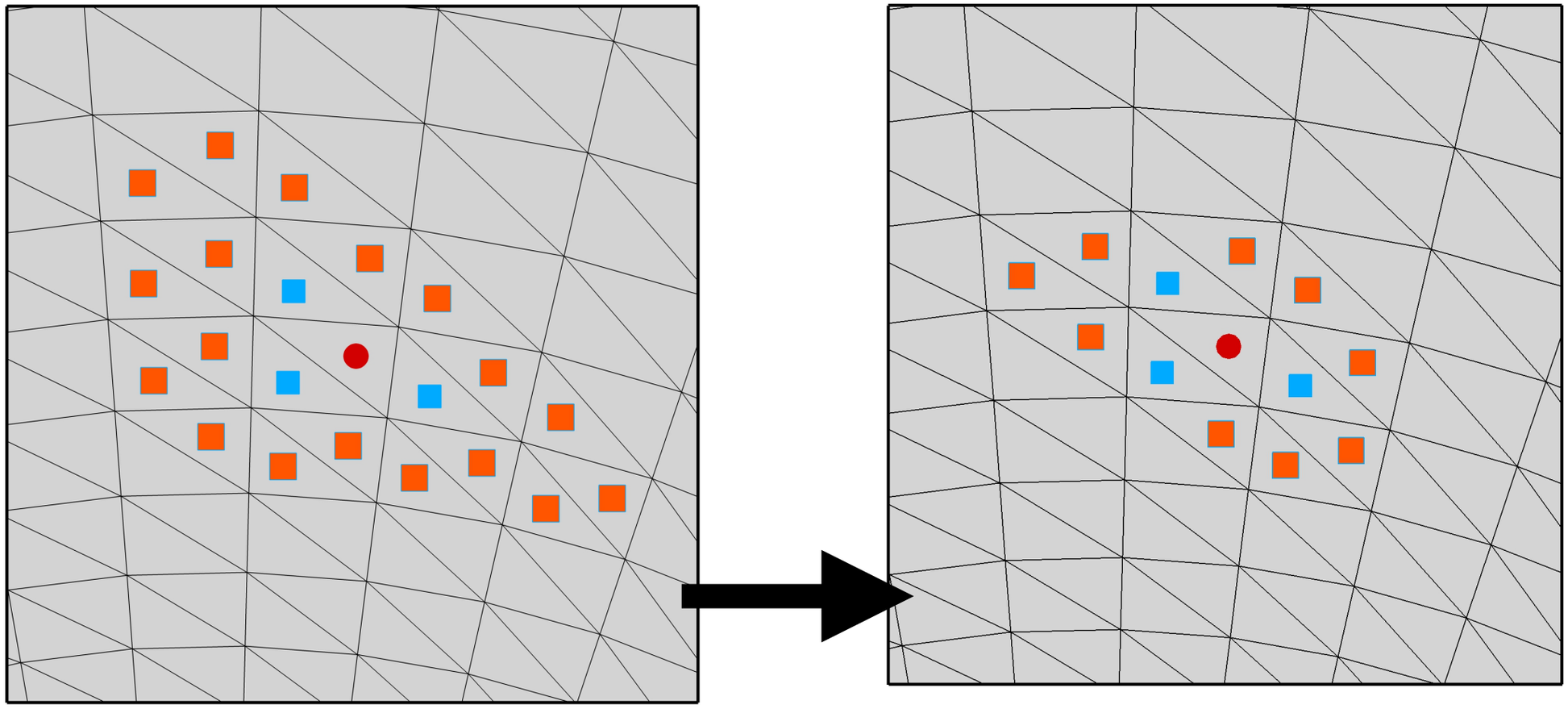


F-ANG : ***F**ace-**A**veraged **N**odal **G**radient*

$$\nabla \mathbf{u}_f = \frac{1}{2}\left(\nabla \mathbf{u}_1 + \nabla \mathbf{u}_3\right)$$

Residual stencil is dramatically reduced

$$\mathbf{Res}_j = \sum_{k \in \{k_j\}} \mathbf{\Phi}_n(\mathbf{u}_L, \mathbf{u}_R) dS$$



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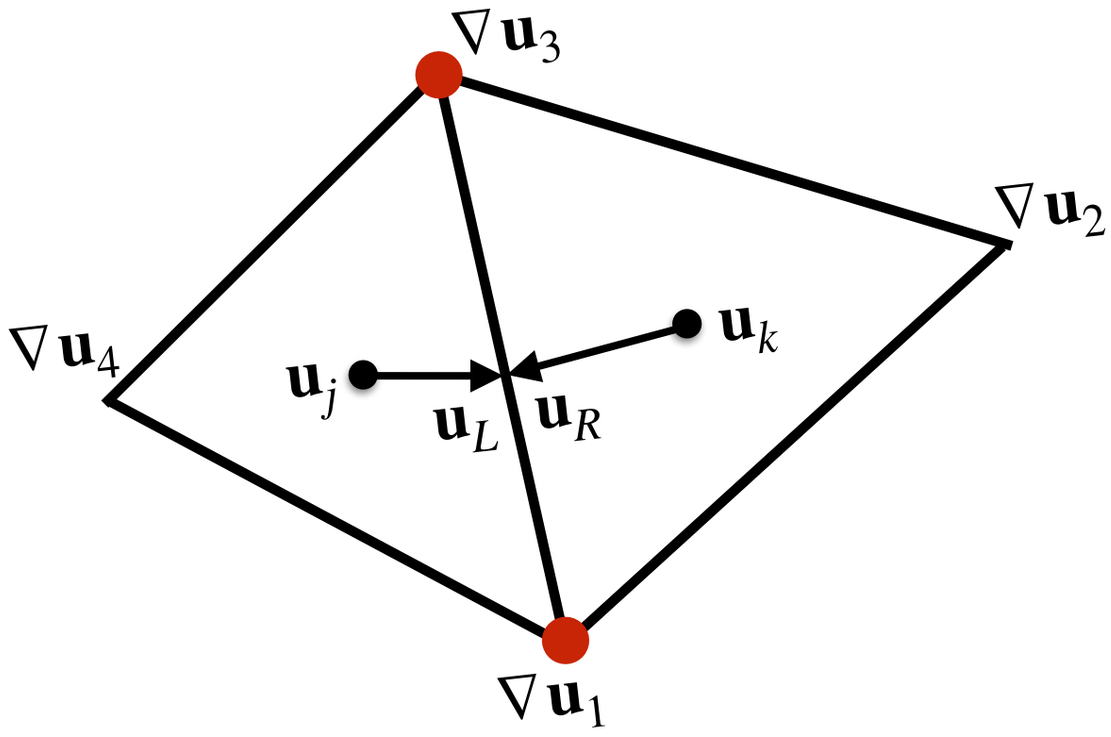
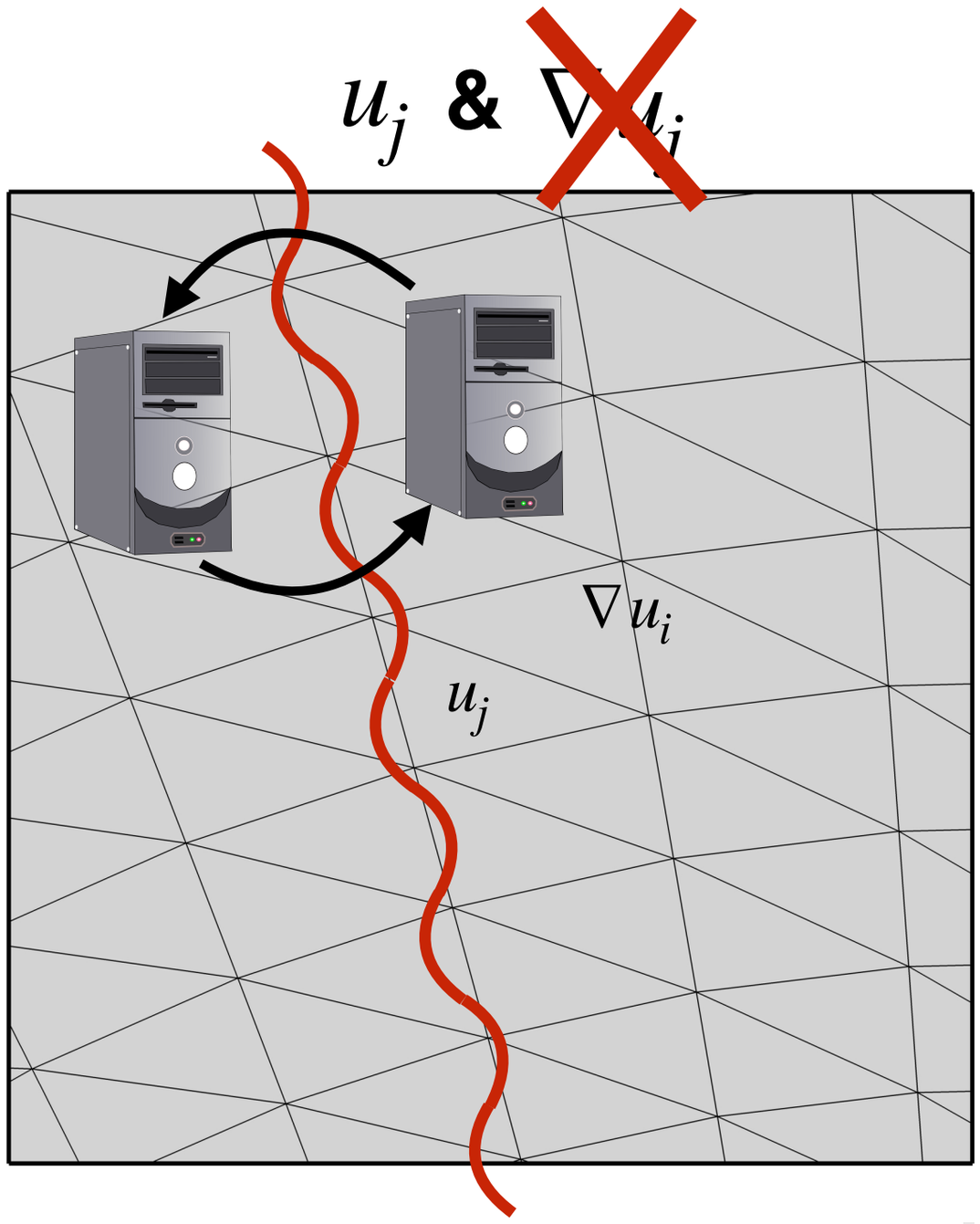
12 !

Communication is not needed for gradients



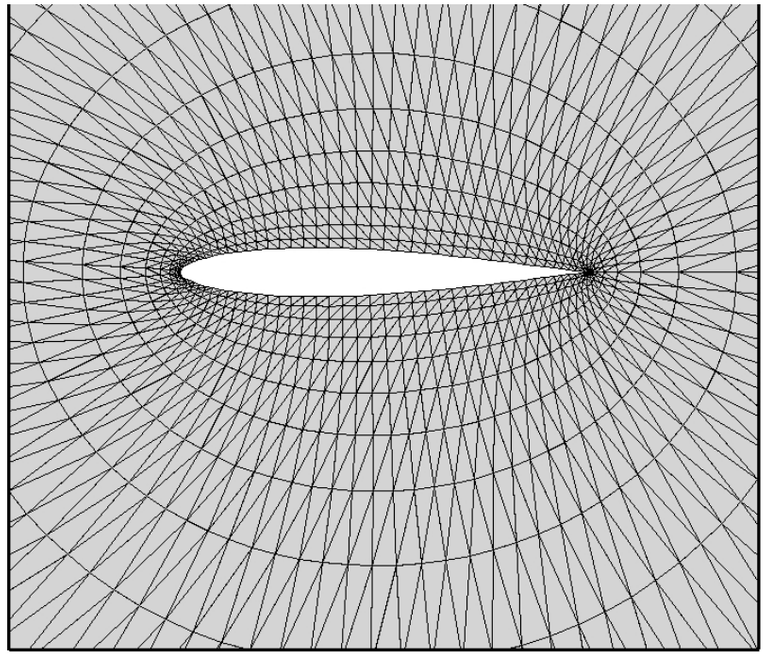


Results

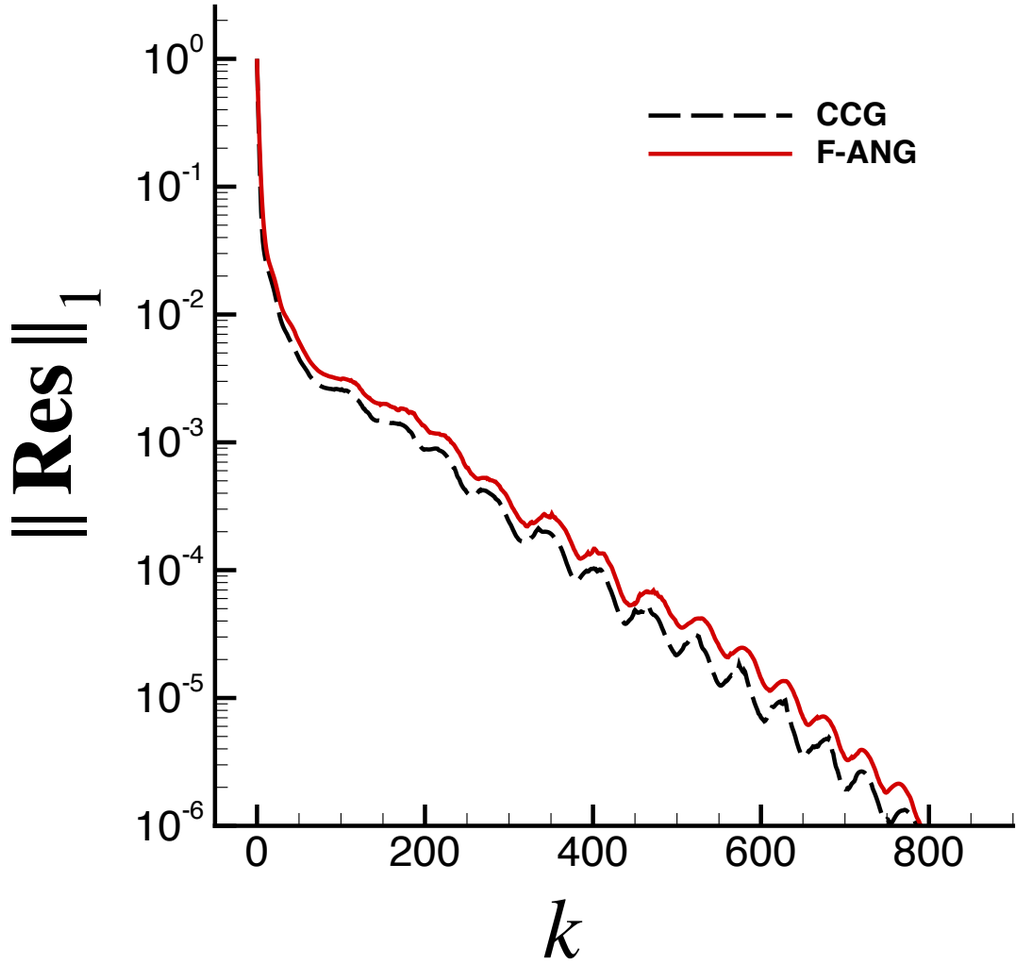
Inviscid: Transonic flow over an airfoil

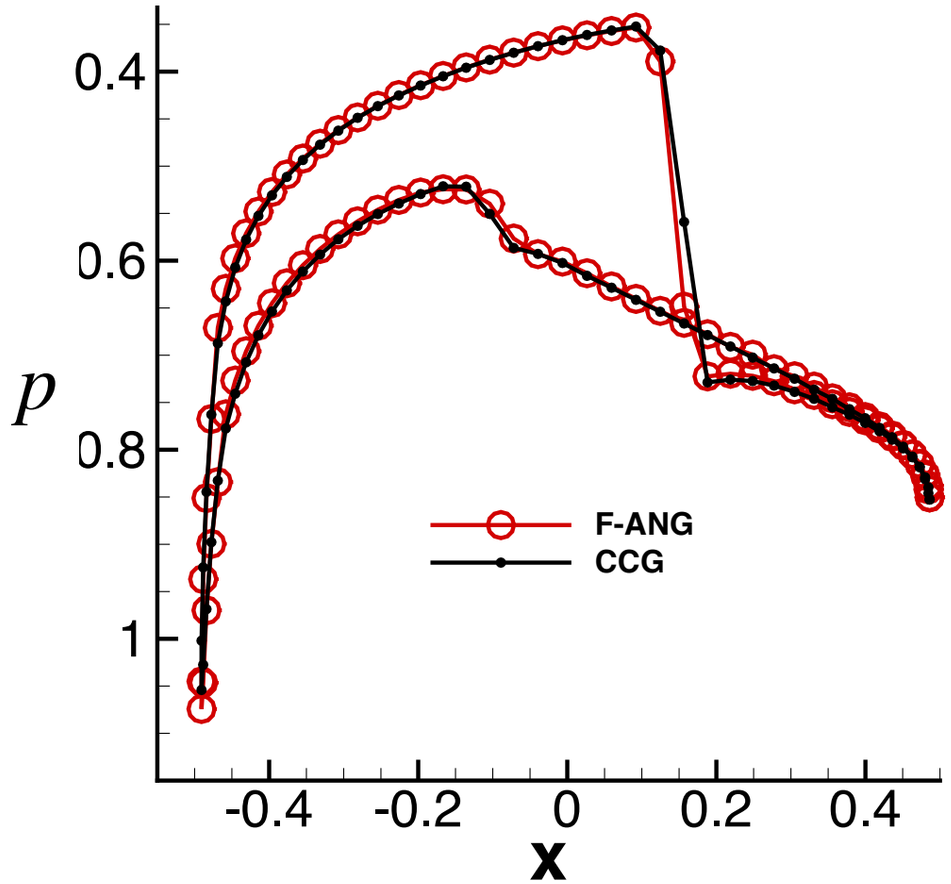
$$M_\infty = 0.85$$

$$\alpha = 1.25$$

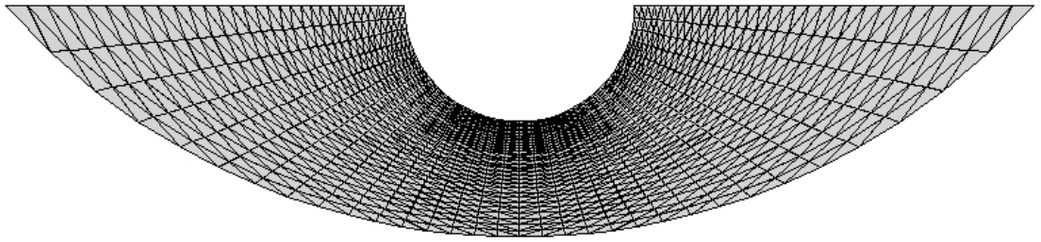


F-ANG works equally well



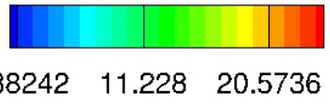


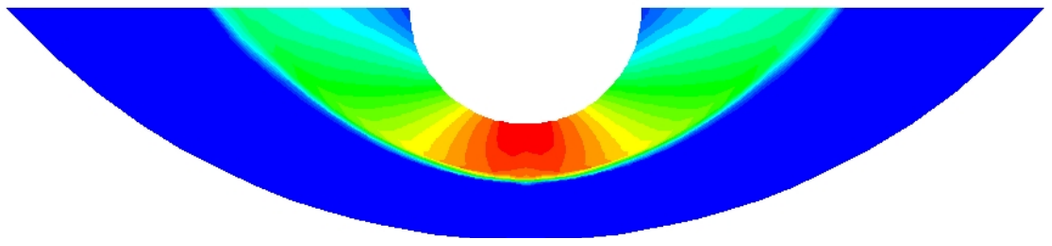
Inviscid: hypersonic flow over a cylinder



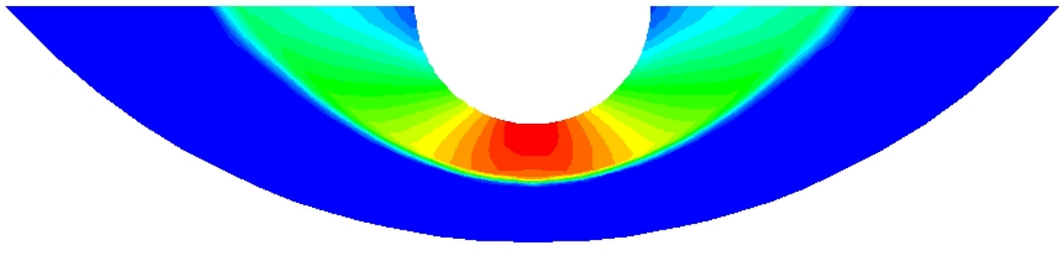
$$M_\infty = 5.2$$

$$\alpha = 0$$

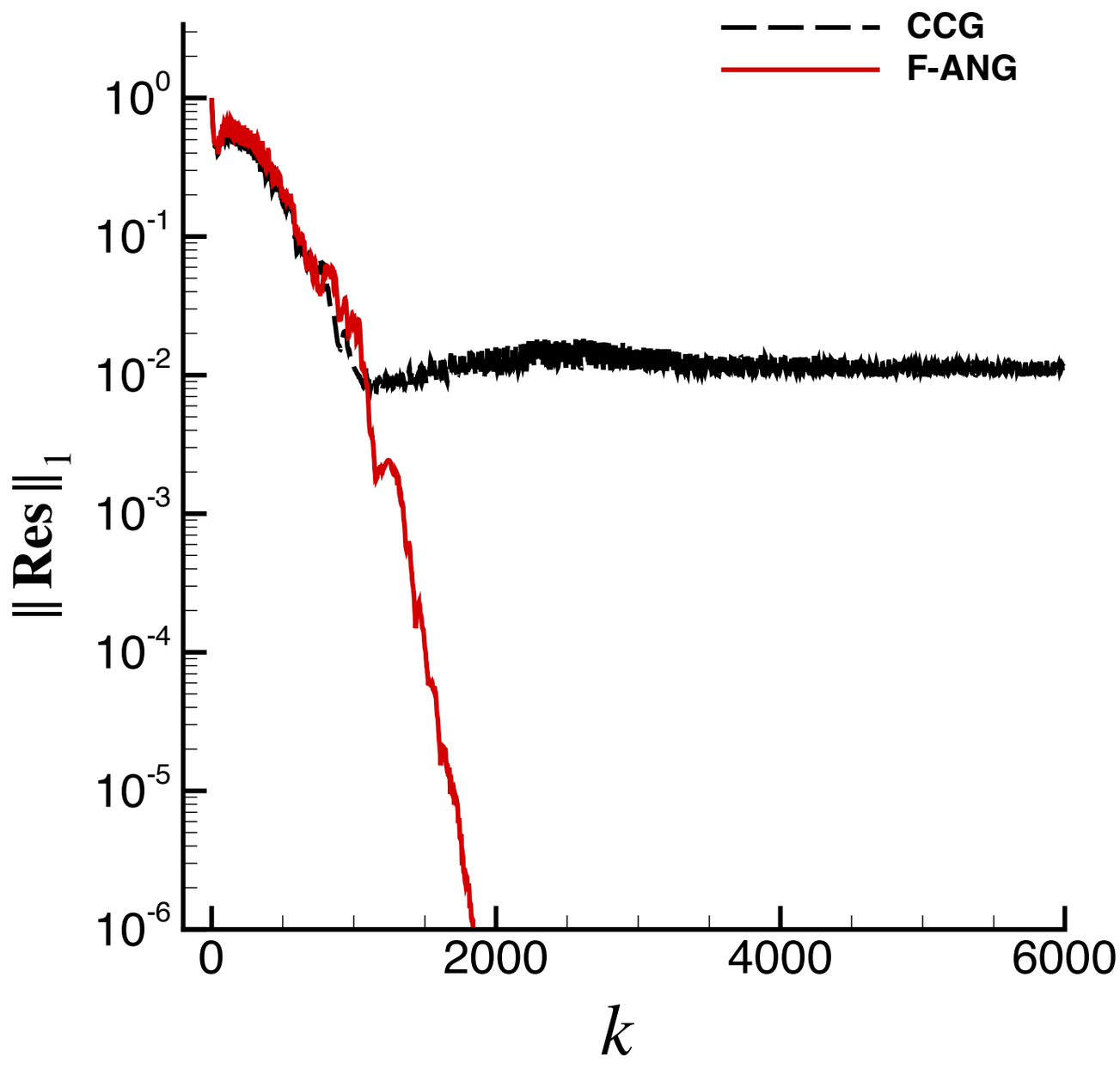




CCG - HLL



F-ANG - HLL

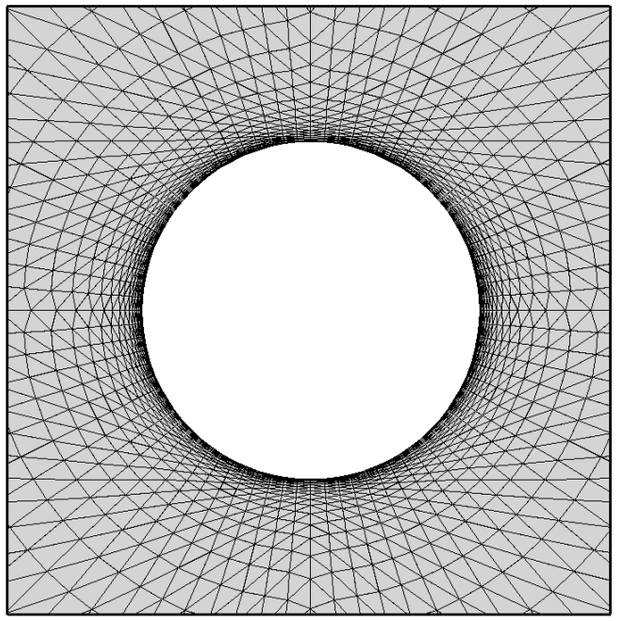


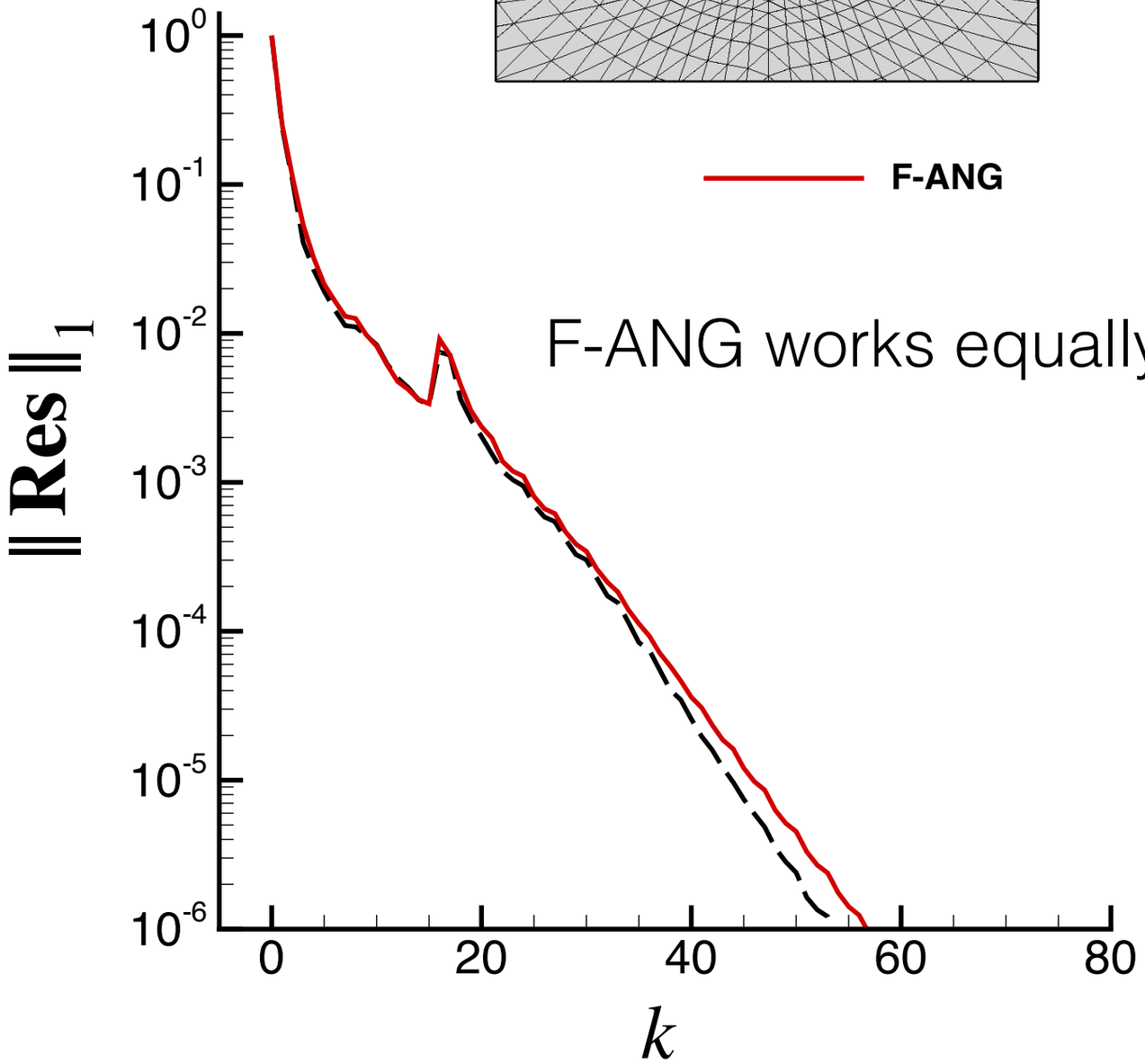
Better convergence with F-ANG

Viscous flow over a cylinder

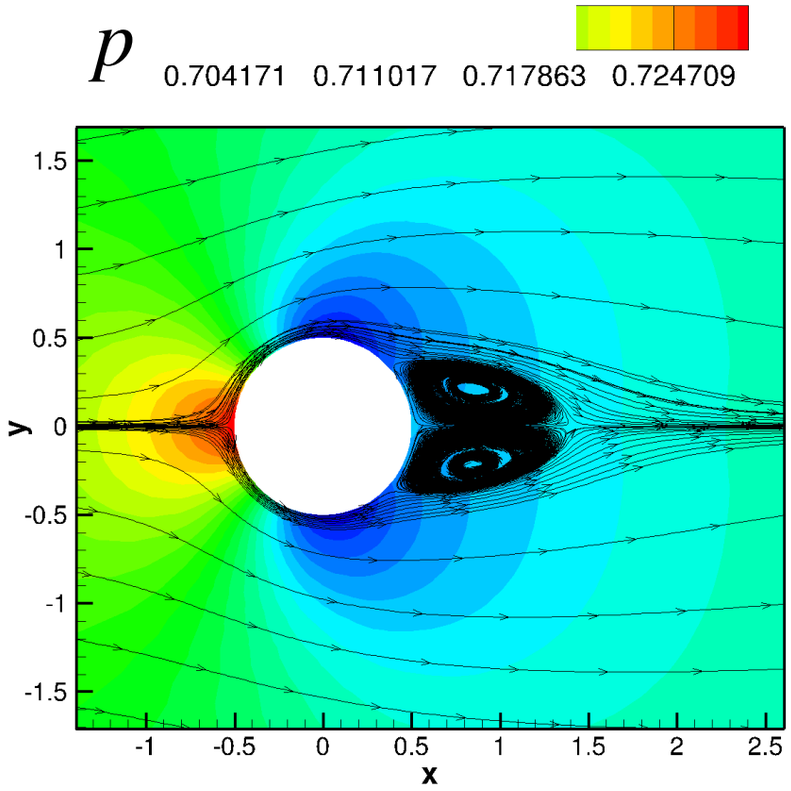
$$M_\infty = 0.3$$

$$Re_\infty = 40$$

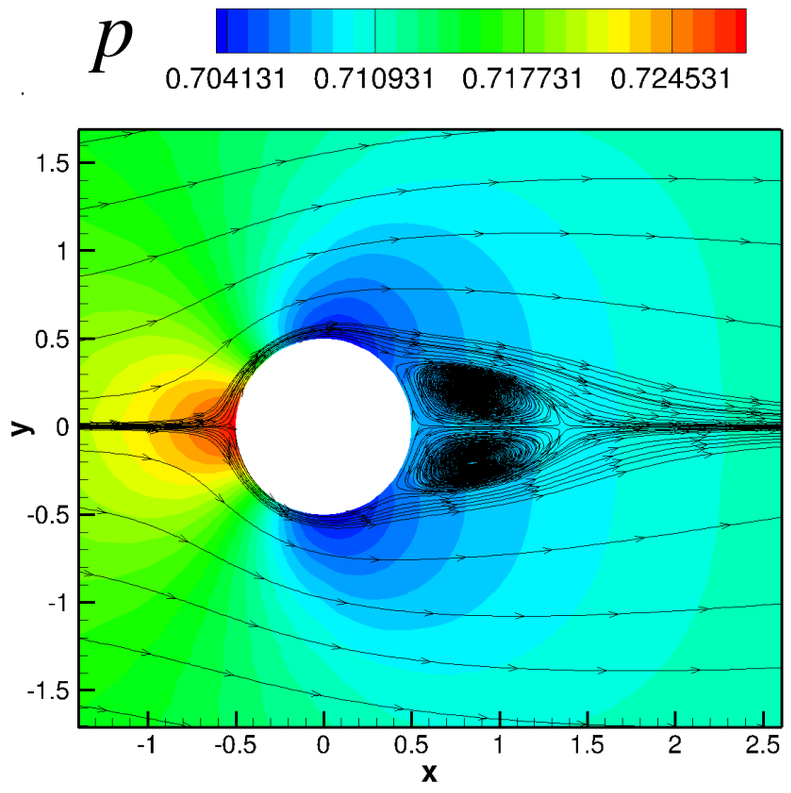




CCG



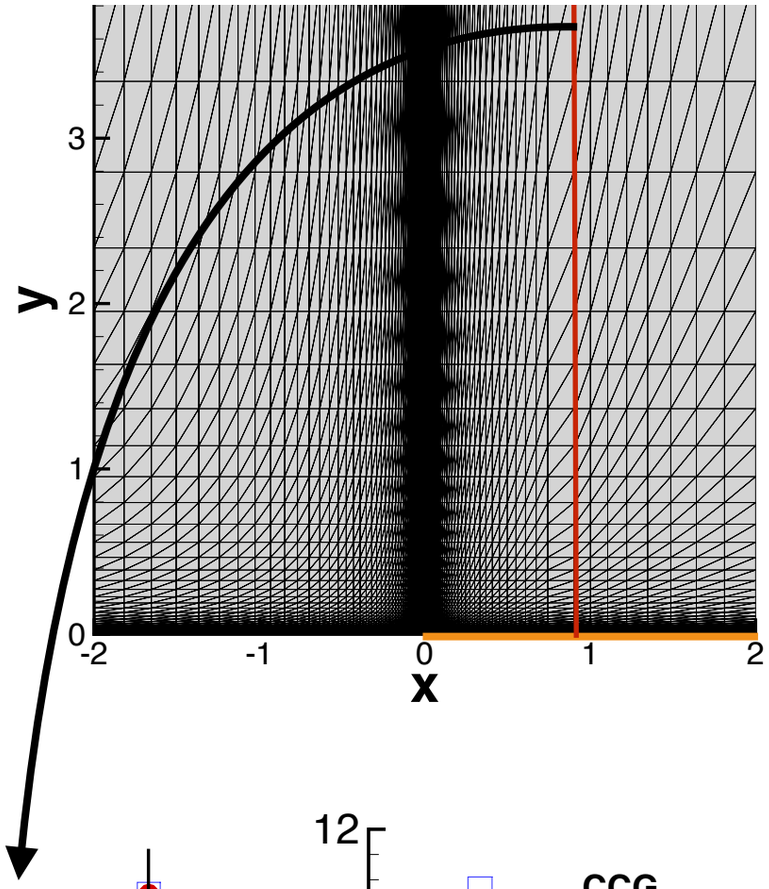
F-ANG

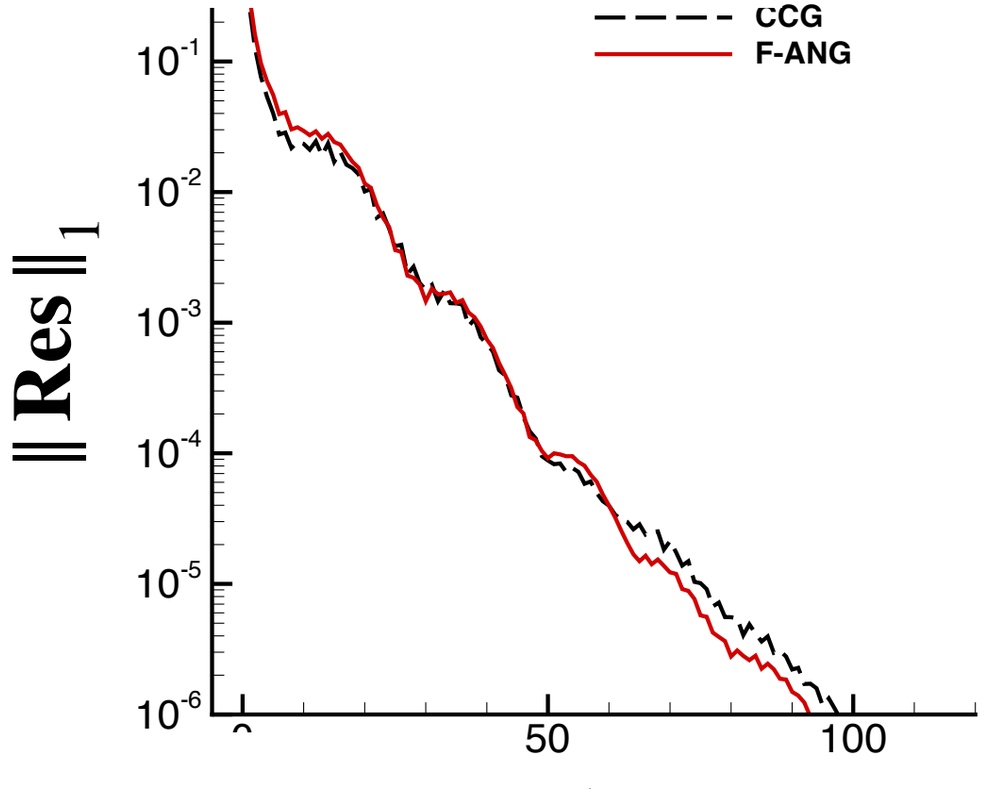


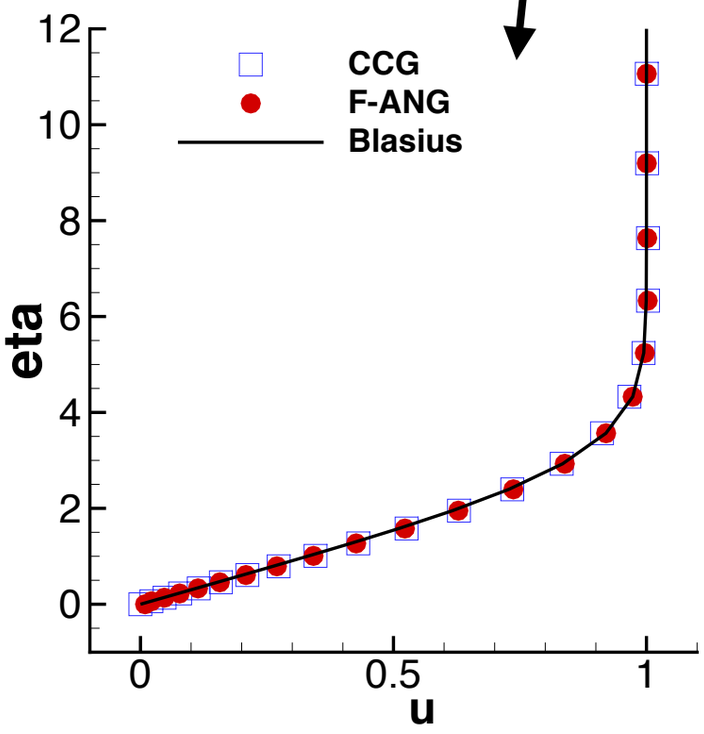
Viscous flow over a flat plate

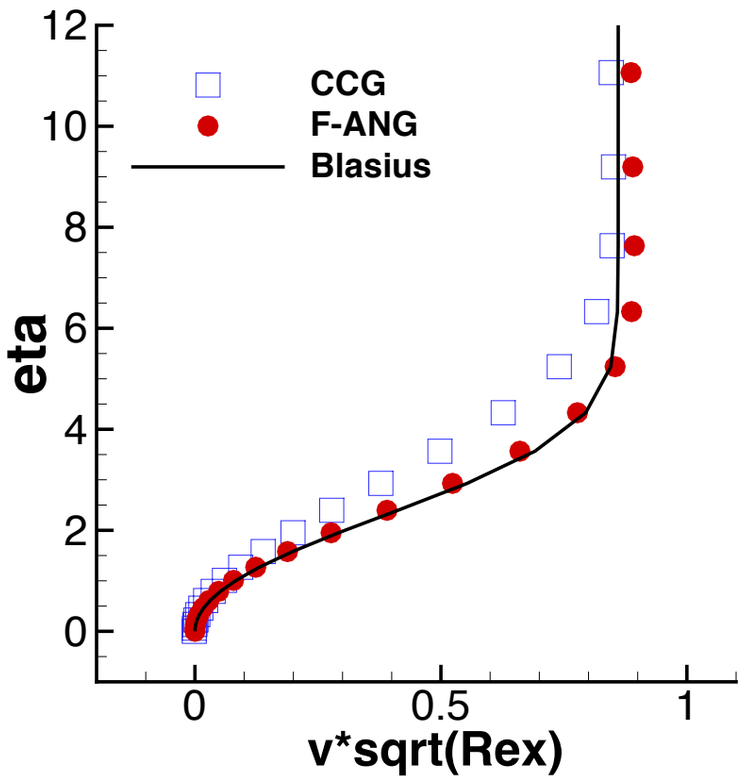
$$M_\infty = 0.15$$

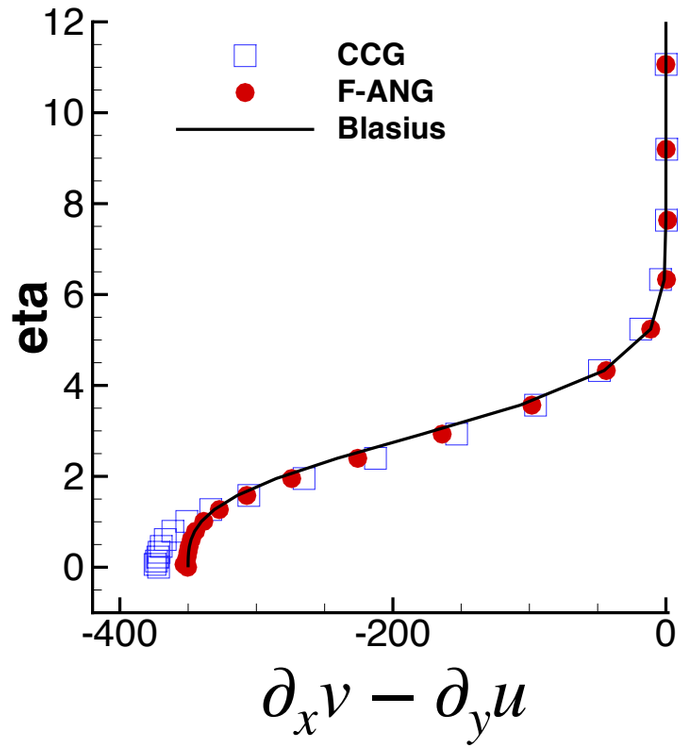
$$Re_\infty = 10^6$$



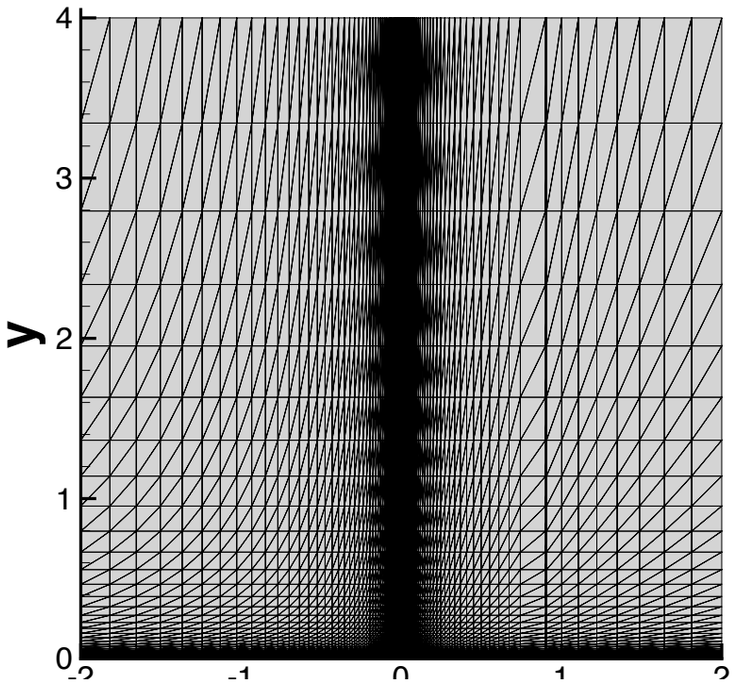








F-ANG converges better and produces more accurate solutions



$[c^x_{jk}, c^y_{jk}]$

	Total	Min	Mean	Max
CCG(vn)	133,554(1.00)	3	11.0	12
F-ANG	34,040(0.25)	3	5.0	6

-75%

$$\frac{\partial \mathbf{Res}}{\partial \mathbf{U}}_{j,k}$$

	Total	Min	Mean	Max
CCG(vn)	243,792(1.00)	7	21.5	22
F-ANG	144,896(0.59)	4	12.8	13

-40%

21.5: x x x x x ...x x x

↓

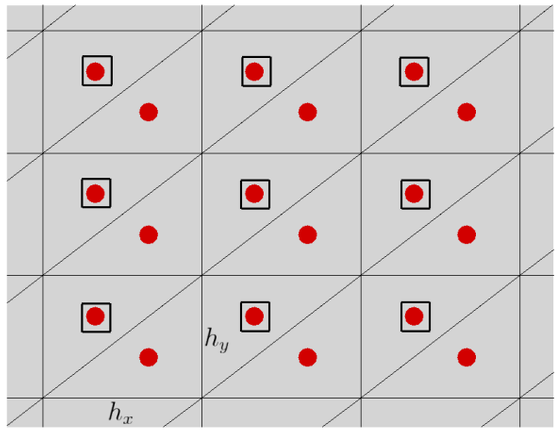
12.8: x x ..x

Cost of gradients and Jacobian size are greatly reduced

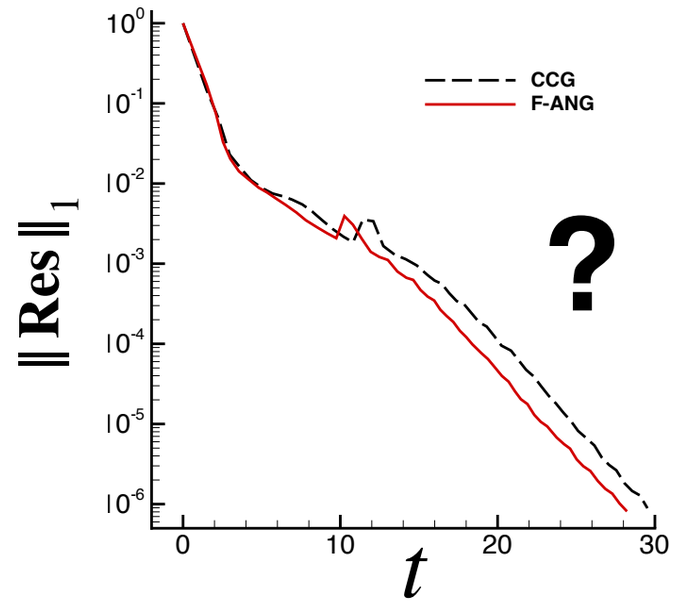
Future work

The work presented is just a preliminary study to investigate its potential; there is much to be done:

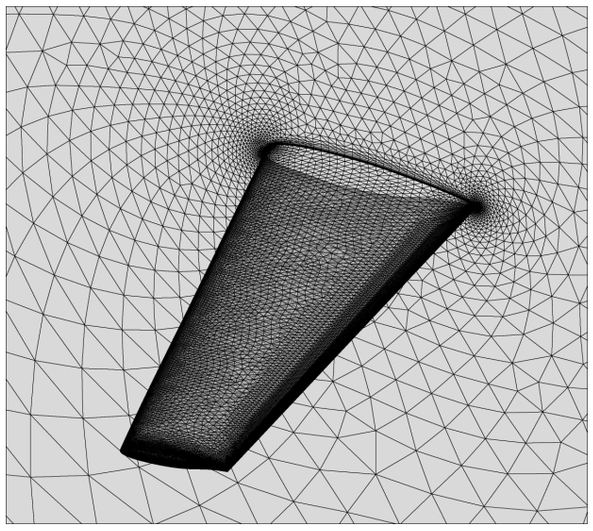
- Proof of convergence
- Actual saving in CPU time

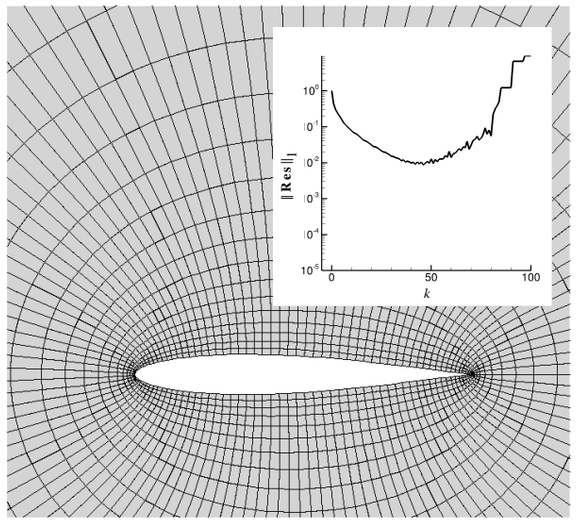


$$\rho(\mathbf{I} - \mathbf{J}^{-1}\mathbf{R}) < 1?$$

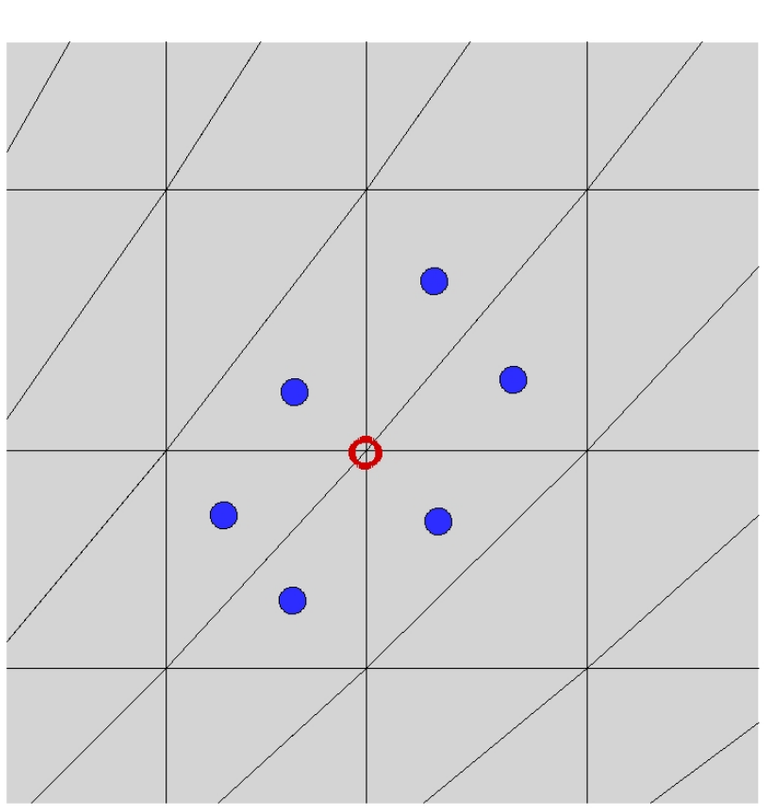
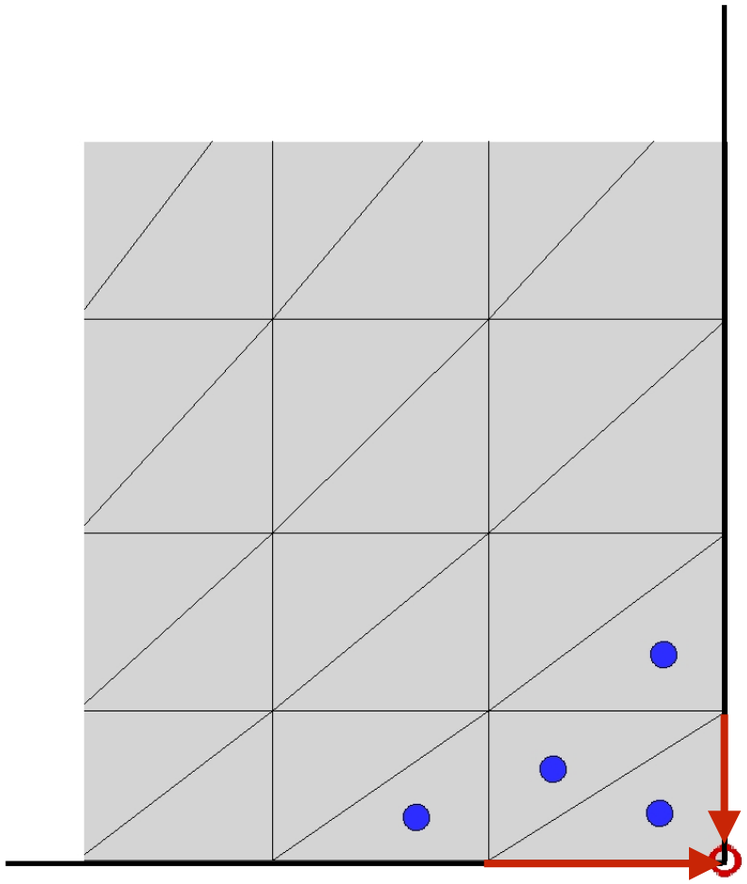
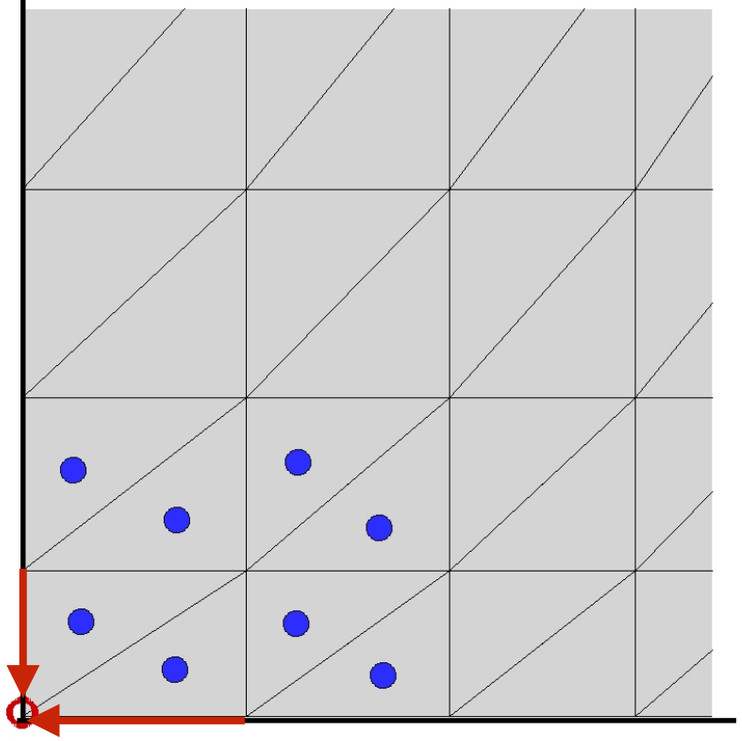


- 3D tetrahedral grids
- Extensions to mixed grids





Extra slide:

$$\begin{bmatrix} u_i \\ \partial_x u_i \\ \partial_y u_i \end{bmatrix} = \sum_k \begin{bmatrix} c_{ij}^x \\ c_{ij}^x \\ c_{ij}^y \end{bmatrix} \bar{u}_j$$

$$nnghbrs \geq 3$$