ON QUANTITATIVE ANALYSIS METHODS FOR MULTIGRID SOLUTIONS

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Abstract. Limitations of the classical local-mode Fourier (LMF) analysis in application to multigrid solutions of variable-coefficient problems are well known. This paper introduces alternative, very general, quantitative analysis methods for multigrid solutions of partial differential equations. The methods are applied to available, nonperfect multigrid solvers that deal with practical problems. The analysis methods considered in this paper focus on the main complementary parts of a multigrid cycle: relaxation and coarse-grid correction. Tasks for these parts, e.g., error smoothing for relaxation and reduction of smooth errors for coarse-grid correction, are assumed to be assigned in advance. Idealized relaxation (IR) and idealized coarse-grid (ICG) iterations are introduced. In these iterations, one part of the cycle (relaxation for IR iterations and coarse-grid correction for ICG iterations) is replaced with an idealized imitation known to be efficient for the assigned task; its complementary part is the actual part of a two-grid cycle. The analysis compares performances of the actual cycle and idealized iterations.

The IR and ICG iterations are very general and can be directly applied in the most complicated simulations including highly variable (or nonlinear) coefficients, complex geometries, and unstructured grids. The results of this analysis are not single-number estimates; they are rather convergence patterns of the iterations that may either confirm or refute expectations indicating what part of the actual solver is inefficient in carrying out the assigned task. The generality of the analysis makes it a valuable tool for analyzing complicated large-scale computational problems, where no other analysis methods are currently available.

In this paper, the IR and ICG iterations are applied to model problems of progressive complexity. The analysis proved sensitive to very delicate details of the actual multigrid cycle, pointing clearly to each instance when a part of the algorithm was inefficient.

Key words. multigrid methods, quantitative analysis, partial differential equations

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1. Introduction. The subject of this paper is new, very general quantitative analysis methods for multigrid solutions of partial differential equation (PDE) problems. Multigrid solvers have proven to be the most efficient iterative methods for discretized PDE systems. Brandt [3, 7] defines a multigrid solver as having textbook multigrid efficiency (TME) if the solutions to the governing system of equations are attained in a total computational work that does not exceed a few (less than 10) minimal work units; a minimal work unit is defined as the computational work required for one residual evaluation. The TME descriptor stems from the efficient multigrid solutions that have been demonstrated for elliptic problems and are available in multigrid textbooks [12, 27, 28]. Recently, TME methods have been developed for several nonelliptic problems in computational fluid dynamics (CFD) [26].

The development of quantitative analysis methodologies for iterative solutions of large-scale computational problems has lagged behind the multigrid practices. Some
methods, such as direct analyses of matrices corresponding to iterative solutions of
discretized differential equations, may be useful for small model problems (see, e.g.,
[13] for the matrix analysis of one-dimensional defect-correction iterations), but are
prohibitively expensive for multidimensional large-scale problems. Other methods,
such as [18, 29], are qualitative and cannot provide sharp convergence estimates for
practical algorithms. For many years, the local-mode Fourier (LMF) analysis has
been the major practical tool for quantitative evaluation of multigrid convergence.
The details of the analysis can be found in the literature [3, 6, 25, 27, 28]. Certain
limitations of the LMF analysis, such as inability to account for boundary conditions
and irregular coefficients, have also been known for a long time. These limitations
make the LMF analysis inapplicable to many important computational problems, such
as problems with highly variable coefficients, complex geometries, and unstructured
grids, where other, more general, analysis techniques must be applied.

A typical approach to analyzing complex problems is based on the following steps:
(1) identify contributors to different aspects of the problem, (2) formulate simplified
models to isolate each identified contributor, (3) solve the simplified models, (4) ex-
tend the solutions to actual problems that cannot be directly analyzed. When applied
to iterative methods, such an approach often leads to dramatic improvements of con-
vergence properties of iterations. However, if the actual difficulty is represented not
by a single object or phenomenon, but rather by interactions between several (or
many) objects and phenomena, analytically solvable models of such interactions are
scarce.

An alternative, more generally applicable approach is to employ an available,
nonperfect solver to deal with a practical problem in its entirety, and then to isolate,
identify, and improve the parts of the solver responsible for the less-than-optimal
performance. This approach closely relates to the compatible relaxation concept in-
troduced in [8] to evaluate multigrid efficiency of a given combination of a relaxation
scheme with a set of coarse-grid variables.

The analysis methods introduced in this paper focus on the main complementary
parts of a multigrid cycle: relaxation and coarse-grid correction. The full-weighting
restriction and bilinear prolongation operators are fixed and assumed suitable for
efficient multigrid solution. The novel analysis techniques, idealized relaxation (IR)
and idealized coarse-grid (ICG) iterations, are applied to two-grid cycles solving PDE
problems. Each part of the cycle is assigned with a task known in advance, e.g.,
relaxation may be assigned to smooth errors, coarse-grid correction may be assigned
to reduce smooth error components.

To apply the new analysis, we first choose a desired sample fine-grid solution
(zero is a natural choice for linear problems) and substitute it into the equations to
generate the corresponding right-hand side and boundary data. Then we form an
initial guess (for example, a random perturbation of the solution); thus, the fine-
grid algebraic error is known. In the proposed analysis, idealized iterations probe
the actual two-grid cycle to identify parts limiting the overall efficiency. In these
iterations, one part of the cycle is actual, and its complementary part is replaced with
an idealized imitation. The idealized imitations do not depend on the operators to be
solved. They are numerical procedures acting directly on the known algebraic error to
efficiently fulfill the task assigned to the corresponding part of the two-grid cycle. The
results of the analysis are not single-number estimates; they are rather convergence
patterns of the iterations that may either confirm or refute our expectations indicating
what part of the actual cycle is not efficient in carrying out the assigned task. The
possible implications from this analysis are at least twofold: one can seek a better
implementation of the identified part of the solver leaving other parts unchanged or one can adjust the algorithmic design by changing assignments to some (or all) parts of the solver. As an illustration of design adjustments, in some multigrid solvers for nonelliptic PDE problems, the relaxation assignment is extended beyond smoothing to reduce some smooth error components as well.

The new analysis methods can be regarded as a numerical extension of the LMF analysis to the problems where the classical LMF analysis is inapplicable. The idealized iterations are very general and can be directly applied in the most complicated situations including highly variable (or nonlinear) coefficients, global solvers, complex geometries, and unstructured grids. Similar to the LMF analysis, the idealized iterations are not constructive; they indicate algorithmic inefficiency but do not suggest a remedy. On the other hand, easily available additional information, such as the frequency content (in the LMF analysis) or shape and patterns (in the IR and ICG analysis) of slowly converging error, may hint at possible improvements to the algorithm. As shown by examples in section 5, the new analysis methods are sensitive to very delicate details of the tested two-grid cycle, clearly indicating what part of the algorithm is inefficient.

The material in this paper is organized as follows. Section 2 introduces the IR and ICG iterations as analysis methods for a two-grid cycle. In section 3, a motivating stagnation-flow problem, which is very difficult for standard analytical methods, is formulated. Section 4 discusses the limitations of LMF analysis in application to variable-coefficient problems. Section 5 presents examples of IR and ICG analysis applications; the limits of applicability of the LMF analysis are also demonstrated. An efficient multigrid solver for the motivating stagnation-flow problem that capitalizes on the findings of the new analysis is reported in section 6. The paper is concluded with some discussions in section 7.

2. IR and ICG iterations. IR and ICG iterations are analysis methods that test computational efficiency of a two-grid cycle. The two-grid cycle amplification matrix, \( M_h \), transforms the initial fine-grid algebraic error, \( e_{\text{old}}^h \), into the after-cycle error, \( e_{\text{new}}^h \).

\[
\begin{align*}
\text{IR:} & \quad e_{\text{new}}^h = M_h e_{\text{old}}^h,
\end{align*}
\]

The amplification matrix can be defined as

\[
M_h = S_h^{\nu_2} C_h S_h^{\nu_1}.
\]

Here, \( \nu_1 \) and \( \nu_2 \) are small nonnegative integers representing the number of pre- and postrelaxation sweeps, \( S_h \) is the relaxation amplification matrix, and \( C_h \) is the amplification matrix of the coarse-grid correction:

\[
C_h = E_h - P_h^H L_{H}^{-1} R_{H}^H L_h,
\]

where \( L_h \) and \( L_H \) are the fine- and coarse-grid operator matrices, \( P_h^H \) and \( R_h^H \) are the prolongation and restriction matrices, and \( E_h \) is the fine-grid identity matrix.

For IR iterations, the coarse-grid correction part is actual and the relaxation is idealized. Assuming that the relaxation assignment is smoothing only, the idealized relaxation may be defined as an explicit error-averaging procedure. Some efficient averaging procedures can be derived from relaxation matrices, e.g., Gauss–Seidel or Jacobi, for a discretized Laplace operator. In this paper, we employ two IR procedures.
The first error-averaging procedure,

\[
e_{i_1,i_2} \leftarrow \frac{1}{4} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} e_{i_1,i_2},
\]

is performed in a lexicographic order and corresponds to the Gauss–Seidel relaxation for the five-point Laplacian. The second IR procedure corresponds to standard full-weighting averaging with stencil

\[
e_{i_1,i_2} \leftarrow \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} e_{i_1,i_2}.
\]

The averaging (2.5) is simultaneously performed at all interior points. This procedure can be considered as a damped Jacobi relaxation for a nine-point Laplacian.

In IR analysis of a two-grid cycle, at each relaxation step, the known exact solution, if not zero, is subtracted from the current approximation to obtain the algebraic error function. The explicit averaging procedure is applied directly to the error function. The exact solution is then added back. The coarse-grid correction remains unchanged. If the convergence of IR iterations is much better than the convergence of the actual cycle, we conclude that the actual relaxation is not a sufficiently good smoother. Slow convergence of IR iterations indicates insufficient coarse-grid correction.

In ICG iterations, the relaxation scheme is actual and coarse-grid correction is idealized. Assuming that the actual restriction and prolongation operators are suitable for efficient multigrid solution, the ICG correction involves ideal operators \(L_H\) and \(L_h\), such that \(L_H^{-1}\) is an accurate approximation to \(L_h^{-1}\) for smooth components. The best ideal operators are the corresponding identity matrices. With this choice, the ICG correction becomes

\[
C_{\text{ideal}} = E_h - P_H R_H^H.
\]

In ICG analysis of a two-grid cycle, the ideal coarse-grid correction replaces the actual one. The idealized \(C_{\text{ideal}}\) is applied directly to the known algebraic errors obtained after pre-relaxation sweep(s) of the actual relaxation. If the performance of the ICG iterations is much better than the performance of actual two-grid cycle, we can conclude that the actual coarse-grid correction is lacking. Slow convergence observed in the ICG iterations is a sign of poor smoothing in relaxation.

The intended application of the IR and ICG iterations is to identify inefficient parts of an unacceptably slow actual cycle. It is assumed that the efficiency goal is defined in advance together with the assignments for each part of the cycle. The overall efficiency considerations set some additional requirements for the idealized imitations: in particular, the “reference cycle,” which uses both IR and ICG iterations, has to meet the efficiency goal. In general, the convergence rate of the reference cycle represents a sensitivity threshold, i.e., the IR and ICG iterations, with given idealized imitations, may suggest some meaningful improvements only for actual cycles with convergence rates significantly slower than this threshold. The numerical tests where actual cycle converges with the rates superior (or comparable) to the threshold should be regarded as indications of sufficient efficiency of the actual cycle.

Note that the idealized iterations are not restricted to structured-grid applications. All multigrid algorithms, including unstructured-grid solvers, have inherently
available averaging procedures as well as restriction and prolongation operators that can be used for the idealized iterations.

3. Motivating problem. The set of two-dimensional nonconservative incompressible Euler equations is defined as

\[
\begin{align*}
  uu_x + vu_y + p &= f_u, \\
  uv_x + vv_y + p &= f_v, \\
  u_x + v_y &= f_c,
\end{align*}
\]

(3.1)

where variables \( u, v, \) and \( p \) represent the \( x \)-directional velocity, the \( y \)-directional velocity, and the pressure, respectively. The first and second equations are referred as the \( u \)- and \( v \)-momentum equations, respectively; the third equation is the continuity equation. Standard boundary conditions for this formulation are the velocity components specified at the inflow boundary, the pressure specified at the outflow boundary, and the tangency condition (where the normal velocity component is set to zero) defined at the body surface.

The plane stagnation flow is defined as a flow against a solid plane located at \( x = 0 \). The exact solutions have the form

\[
\begin{align*}
  u &= -ax, \\
  v &= ay, \\
  p &= -a^2 \frac{x^2 + y^2}{2},
\end{align*}
\]

(3.2)

where \( a \) is an arbitrary constant.

A typical computational domain for the plane stagnation flow is a square \( (x, y) \in [-D, 0] \times [-D/2, D/2] \). The inflow boundary is defined at the left vertical edge, \( x = -D \), the outflow boundaries are at the horizontal edges, \( y = \pm D/2 \), and the tangency condition is defined at the plane surface, \( x = 0 \). If the boundary conditions are symmetric with respect to the axis \( y = 0 \), the stagnation point is located at \( (x, y) = (0, 0) \).

The pressure-equation formulation \([22, 23]\) is obtained from (3.1) by replacing the continuity equation with an equation for the pressure

\[
\begin{align*}
  uu_x + vu_y + p_x &= f_u, \\
  uv_x + vv_y + p_y &= f_v, \\
  (u_x)^2 + 2uv_x + (v_y)^2 + \Delta p &= f_p.
\end{align*}
\]

(3.3)

The major advantage of this formulation is an easily available \( h \)-elliptic collocated-grid discretization. (See definition and discussions related to \( h \)-ellipticity in \([2, 3, 27]\).) Besides the standard set of boundary conditions (given above), the pressure-equation formulation (3.3) requires an additional boundary condition at the inflow boundary. To ensure the same solution for the problems (3.1) and (3.3), this additional boundary condition is taken as the continuity equation enforced at the inflow boundary. This boundary condition can be implemented in a number of ways; a convenient choice is through the Neumann condition for the pressure derivative obtained by manipulating the normal (to the boundary) momentum equation and the continuity equation at the inflow boundary. That is,

\[
\partial_x p = f_u - u\partial_x u - v\partial_y u = f_u + u(\partial_y v - f_c) - v\partial_y u,
\]

(3.4)

where \( f_u + u(\partial_y v - f_c) - v\partial_y u \) is a known quantity at the inflow boundary.
The second-order accurate interior discretization, \( R(q) \), of (3.3) is defined as

\[
R(q) = \frac{1}{2} \left( \begin{array}{c}
\partial_x (u_0 + v_0) \\
\partial_y (u_0 + v_0) \\
(\partial_x u_0 + \partial_y u_0) + \Delta^h p
\end{array} \right) = \left( \begin{array}{c}
f_x \\
f_y \\
f_p
\end{array} \right),
\]

where \( q = (u, v, p)^T \), and superscripts \( u \) and \( c \) denote second-order accurate upwind and central differences for the first derivatives, respectively. Discretization of the Dirichlet conditions of given velocity components at the inflow boundary and given pressure at the outflow is straightforward. The Neumann conditions for the pressure can be implemented explicitly by employing a one-sided difference approximation for the normal derivative. With this implementation, the discrete pressure equation is not defined at the boundary. An alternative implementation uses two discrete equations for pressure defined at the same boundary point: a central approximation for the normal pressure derivative and a modified pressure equation. The pressure value located in the exterior of the computational domain is eliminated from these two equations. In the modified pressure equation, some velocity derivatives, \( u_y, v_y, u_x = f_c - v_y \), are computed analytically from the Dirichlet conditions for the velocities; the derivative \( v_x \) is discretized with a one-sided stencil. An advantage of the latter implementation is a more compact stencil obtained for differencing the pressure at the boundary. For the discrete formulation (3.5), two numerical closure equations are required at each outflow boundary and at the plane surface. The closure equations are usually modified discrete approximations to the target differential equations different from the discretizations in the interior. Detailed discussions and numerical tests for different boundary conditions can be found in [17].

The Newton linearization, \( \frac{\partial R}{\partial q} \), of the operator (3.5) around the solution \( q \) is defined as

\[
\frac{\partial R}{\partial q} = \left( \begin{array}{ccc}
Q_x + (\partial_x u) \\
2(\partial_x^2 u) \\
2(\partial_x u)(\partial_y u) + 2(\partial_y^2 u) + \Delta^h p
\end{array} \right)
\]

where \( Q_x = u\partial_x^2 + v\partial_y^2 \) is a second-order accurate upwind discretization of the linear convection operator.

4. Limitations of LMF analysis. In this section, we discuss some limitations of the LMF analysis in applications to variable-coefficient problems. A rigorous LMF analysis for uniformly elliptic variable-coefficient problems has been considered in [5]. The classical LMF analysis applied to a variable-coefficient problem is defined in the following three steps.

1. Form a set of constant-coefficient approximations by freezing the variable coefficients at some locations, e.g., grid nodes.
2. Analyze each constant-coefficient problem on the infinite (or periodic) domain.
3. Take the final estimate of the analysis as the worst estimate among all the constant-coefficient problems.
The infinite grid assumption makes the LMF analysis intrinsically incapable of predicting the effects of boundary conditions. On finite computational domains, the choice of discrete boundary conditions and the way these conditions are treated in iterations can dramatically affect the convergence properties of iterations. A poor choice of boundary conditions may lead to an ill-posed discrete problem, even if the corresponding differential problem is well posed. Significant convergence deterioration may occur in the simplest well-known problems, e.g., the Laplace equation, as a result of improper handling of the boundary conditions. Some modifications of the LMF analysis, such as a half-space analysis of the first differential approximation (FDA analysis) [1, 11] and a discrete half-space analysis [16], have been developed to take the boundary conditions into account.

For the same reason, the LMF analysis cannot account for the downstream accuracy propagation that is an important convergence mechanism for some nonelliptic problems. In the presence of this mechanism, the accuracy is first achieved near the inflow boundary and then propagates downstream into the interior of the domain. (For illustration, see Figure 5.3 in section 5.2.) The computational domain affected by the downstream accuracy propagation may become very large, growing with each iteration. In the case of a downstream marching, it may expand to the entire domain in just one iteration. The estimates of the LMF analysis for nonelliptic flow problems can, however, be accurate in the areas that have yet to be affected by the downstream accuracy propagation from the inflow boundary. For iterative solvers of hyperbolic problems, e.g., nonmarching multigrid methods for the convection equation, LMF analysis estimates are usually accurate for the initial convergence; eventually, the accuracy propagated downstream from the inflow boundary extends over the entire computational domain.

Another serious limitation of the LMF analysis is the constant-coefficient framework itself. Accuracy of the LMF analysis extension to general variable-coefficient (or nonlinear) problems is limited by the accuracy of constant-coefficient approximations to the target problem. In this context, approximation means that the values of actual (variable) coefficients do not deviate significantly from a constant state.

There are many ways to measure how well a constant-coefficient formulation approximates a variable-coefficient one in a certain region. Here, we adopt a strong requirement that limits variation in each variable coefficient separately. Let \( C_k(x) \), \( k = 1, 2, \ldots \), be variable coefficients of the operator \( L_v \), and let

\[
C_k^0 = C_k(x_0)
\]

be values of \( C_k(x) \) frozen at the point \( x_0 = (x_0, y_0) \). We regard the constant-coefficient operator \( L_c \) obtained by freezing coefficients of \( L_v \) as approximating \( L_v \) in the vicinity of the point \( x_0 \) with a relative accuracy \( \epsilon \), if there is a nonvanishing neighborhood \( \Omega_{x_0}^{s_1, s_2} = \{ x = (x, y), |x - x_0| < s_1, |y - y_0| < s_2 \} \) such that, within this neighborhood,

\[
|C_k^0 - C_k(x)| < \epsilon |C_k^0|
\]

for any coefficient \( C_k(x) \) of \( L_v \). The rectangular domain \( \Omega_{x_0}^{s_1, s_2} \) is called the \( \epsilon \)-neighborhood of the point \( x_0 \).

For any chosen \( \epsilon \), the values of \( s_1 \) and \( s_2 \) provide the scales at which the constant-coefficient problem approximates the variable-coefficient one with relative \( \epsilon \)-accuracy. Reasonable values for \( \epsilon \) are between zero and one, with \( \epsilon = 0.5 \) intuitively considered as a practical value. See Figure 4.1 for illustration of typical \( \epsilon \)-neighborhoods for a
variable-coefficient convection equation

\[ -xu_x + yu_y = 0. \] (4.3)

For the variable coefficients frozen at the point \( x_0 = (x_0, y_0) \), one should confine the LMF analysis to a computational subdomain that is located entirely within the \( \epsilon \)-neighborhood of the point \( x_0 \). If an \( \epsilon \)-neighborhood is smaller than several mesh sizes in each dimension, the \( \epsilon \)-neighborhood is considered vanishing; the LMF analysis should not be applied at this neighborhood. Implementation of the \( \epsilon \)-neighborhood concept together with an assumption on the minimal number of mesh intervals sets some lower bounds on the ratios of coefficient to mesh size, e.g., the bound \( \frac{h}{\epsilon} > 4 \) occurs for (4.3) with \( \epsilon = 0.5 \) and a minimum of four mesh intervals in the \( y \)-direction per \( \epsilon \)-neighborhood. Within this \( \epsilon \)-neighborhood concept, some regions, such as the stagnation stream line, are inaccessible to the LMF analysis.

5. **Examples of idealized iteration analysis.**

5.1. **Example 1: Skew discretization of Laplace equation.** The first model problem is a two-grid cycle for a skew discretization of the Laplace operator. The computational domain is a unit square with the \( 17 \times 17 \) uniform isotropic Cartesian grid. The discretization

\[ Lu_{i_1, i_2} \equiv \frac{1}{2h^2} \left( u_{i_1+1,i_2+1} + u_{i_1-1,i_2+1} + u_{i_1+1,i_2-1} + u_{i_1-1,i_2-1} - 4u_{i_1,i_2} \right) = 0 \] (5.1)

is subject to the Dirichlet boundary conditions. This formulation lacks \( h \)-ellipticity; therefore, a typical local relaxation exhibits poor smoothing properties. The tested solver is a two-grid (1, 2)-cycle with lexicographic point Gauss–Seidel relaxation, full coarsening, bilinear prolongation, and full-weighting residual restriction.
Classical LMF analysis can easily identify the problem. The smoothing factor, $\mu$, of the Gauss–Seidel relaxation is computed as

$$\mu = \max_{\theta} \frac{e^{i(\theta_x+\theta_y)} + e^{i(\theta_x-\theta_y)}}{4 - e^{-i(\theta_x+\theta_y)} - e^{-i(\theta_x-\theta_y)}},$$

where the maximum is taken over high-frequency modes

$$\theta_{hf} = \left\{ (\theta_x, \theta_y), |\theta_x| \leq \pi, |\theta_y| \leq \pi, \max(|\theta_x|, |\theta_y|) \geq \frac{\pi}{2} \right\}.$$

The smoothing factor approaches unity for $|\theta_x| = |\theta_y| = \pi$.

![Fig. 5.1. Convergence rates for skew Laplacian.](image)

The idealized iterations are effective indicators as well. The averaging procedure used in the IR iterations is given by (2.4). The convergence rates per iteration for actual cycle, reference cycle, IR, and ICG iterations are shown in Figure 5.1. The convergence rate of the two-grid cycle is well above the sensitivity threshold defined by the reference cycle. Recall that reference cycle employs both IR and ICG correction. Both IR and ICG iterations clearly point to poor smoothing in the actual cycle.

Regarding possible convergence improvements, the best advice is to avoid using non-$h$-elliptic discretization schemes. If one must solve the skew-Laplacian formulation, a good strategy is to consider this formulations as two uncoupled problems. Using a checkerboard analogy, the solver for the equations defined at black nodes should be independent of the solver for the formulation at white nodes. Considered at nodes of the same color, the discretization (5.1) is $h$-elliptic.

5.2. Example 2: Convection equation. In this section, we analyze a two-grid solver for the standard first-order accurate upwind discretization of the constant-coefficient convection equation

$$\partial_x^h u + \partial_y^h u = 0,$$

where $\partial_x^h$ and $\partial_y^h$ are two-point upwind differences for the $x$- and $y$-derivatives, respectively. A rectangular computational domain is covered by a uniform Cartesian
grid with the characteristics aligning with the diagonal direction. This is a nonelliptic problem known to represent difficulties for standard multigrid solvers. The solver to be tested is a standard full-coarsening two-grid \((1,2)\)-cycle including a red-black relaxation scheme with an underrelaxation parameter \(w = 0.8\), full-weighting residual restriction, and bilinear prolongation. The coarse-grid discretization is identical to the discretization on the fine-grid. The LMF analysis for this problem is available in [27]. The relaxation smoothing factor is predicted as 0.52; the convergence of the two-grid cycle is limited by coarse-grid correction and is estimated as 0.5 per cycle. The slowly converging error components are smooth components that are much smoother in the characteristic direction than in other directions. These components are called characteristic components.

![Convergence rates](image)

**Fig. 5.2.** Convergence rates for a first-order discretization of the convection equation with periodic boundary conditions in the \(y\)-direction. Standard coarse-grid discretization.

The numerical tests including the two-grid \((1,2)\)-cycle, IR and ICG iterations, and reference cycle have been performed on uniform grid, \(1025 \times 65\) (domain \(16 \times 1\)), with the zero inflow boundary condition at \(x = 0\) and periodic boundary conditions in the \(y\)-direction. The \(x\)-size of the domain has been chosen to be long enough for observing convergence slowdown caused by characteristic error components. The initial approximation is random. The results are shown in Figure 5.2. The averaging procedure used in IR iterations is given by (2.5).

Convergence rates of the two-grid \((1,2)\)-cycle are much slower than the rates of the reference cycle and close to the predictions of the LMF analysis. Convergence rates of IR and ICG iterations identify the coarse-grid correction as a part of the cycle limiting the overall efficiency. Similar to the frequency content in the LMF analysis, the profile of the algebraic errors after many iterations (see Figure 5.3 for errors after 20 and 40 iterations) clearly indicates the type of error that is slow to converge, namely smooth characteristic error. Moreover, the error profile demonstrates the effect of accuracy propagation from the inflow boundary. This important convergence mechanism for nonelliptic problems eludes the LMF analysis.

To improve convergence rates of multigrid solvers for convection-dominated (and other nonelliptic) problems, one can facilitate the accuracy propagation from the inflow boundary by employing a “global” relaxation scheme, e.g., downstream marching \([11, 16]\) or implicit line relaxation \([20]\). In this context, “global” relaxation means that the relaxation scheme strongly affects at least some smooth errors, as opposed
to “local” relaxation that mostly affects high-frequency errors. Within such an approach, the relaxation assignment is extended to reduce both oscillatory and characteristic errors. The approach requiring characteristic error reduction in relaxation can be efficient as long as the characteristics of the nonelliptic operator are not closed. For problems with closed characteristics, the characteristic errors should be reduced in coarse-grid correction.

For efficient coarse-grid correction, the coarse-grid discretization should be adjusted to better approximate the fine-grid operator. One way to adjust the discretization is to employ a Galerkin coarse-grid operator with intergrid transfers satisfying certain accuracy conditions [30]. Another way is to explicitly add some adjustment terms to the coarse-grid operator to match the fine-grid (characteristic-component) first differential approximation [9, 14, 15, 19]. For the fine-grid discretization (5.4), the coarse-grid operator can be modified as

\[
\bar{\partial}_x^h = \partial_x^h + \frac{h}{4} \partial_{xx}^u,
\]

where \(\bar{\partial}_x^h\) is a modified discretization of the \(x\)-derivative, \(\partial_x^h\) is the first-order upwind discretization, and \(\partial_{xx}^u\) is the first-order accurate three-point upwind discretization of the second derivative. The discretization of the \(y\)-derivative has been modified accordingly. With this modified coarse-grid operator, convergence rates of the two-grid (1, 2)-cycle are improved significantly to 0.27 per cycle, as shown in Figure 5.4; the rates of IR iterations also improved. ICG iterations and reference cycle are unaffected, as expected.

5.3. Example 3: Laplace equation with Neumann boundary condition.

In this section, the IR and ICG analyses are applied to the Laplace equation defined on a square domain \((x, y) \in [0, 1] \times [0, 1]\) with one Neumann boundary condition at \(x = 1\) and the Dirichlet conditions at other boundaries. The Laplacian is discretized on a uniform 65 \(\times\) 65 grid with the standard five-point stencil. The Neumann boundary condition is discretized with a three-point, one-sided differencing. The solution method is a full-coarsening two-grid (1, 2)-cycle with lexicographic point Gauss–Seidel relaxation, full-weighting residual restriction, and bilinear prolongation. At the end of the relaxation sweep, the residuals of the Neumann equations at the boundary are zeros.

Fig. 5.3. Error surface after 20 (left) and 40 (right) two-grid (1, 2)-cycles for a first-order discretization of the convection equation with periodic boundary conditions in the \(y\)-direction. Grid 1025 \(\times\) 65.
Fig. 5.4. Convergence rates for a first-order discretization of the convection equation with periodic boundary conditions in the y-direction. Grid $1025 \times 65$. Modified coarse-grid discretization.

Fig. 5.5. Convergence rates for the Laplace equation with Neumann boundary conditions. No boundary relaxation.

For the two-grid $(1, 2)$-cycle, the LMF analysis predicts the convergence rate of 0.12 per cycle. Figure 5.5 shows the residual convergence rates for the two-grid $(1, 2)$-cycle and for the corresponding IR and ICG iterations. IR iterations employ the averaging procedure (2.4) in the interior performed in the lexicographic order. At the Neumann boundary, the error is averaged as

$$e_{i_1,i_2} \leftarrow \frac{1}{2} (e_{i_1,i_2+1} + e_{i_1,i_2-1}).$$

In this test, convergence rates of reference cycle and ICG iterations are identical. The asymptotic convergence rate of the actual two-grid cycle is 0.20, noticeably worse than the rate predicted by the LMF analysis and observed in computations with all-Dirichlet boundary conditions.

The difficulty associated with the Neumann boundary relates to large interior residuals created after relaxing boundary conditions. The standard full-weighting
residual restriction near the Neumann boundary is inaccurate and degrades the quality of the coarse-grid correction. The LMF analysis cannot account for this effect. The ICG and IR iterations clearly indicate that the coarse-grid correction should be improved to accelerate convergence of the two-grid cycle.

There are several approaches to overcome this convergence slowdown; e.g., one can design a more accurate restriction operator near the Neumann boundary [27] or a local relaxation scheme that does not create large interior residuals. A more general approach, directly extendible to more complicated computational problems, is to apply a block relaxation near the boundary relaxing simultaneously all the equations defined at a certain number \( \text{depth} \) of grid lines adjacent to the Neumann boundary. With (almost) zero residuals in a boundary vicinity, an accurate residual restriction is no longer important. We refer to this special block relaxation as boundary relaxation. The boundary relaxation procedure, performed once per cycle before transferring residuals to the coarse grid, can be considered a part of an improved coarse-grid correction scheme. Figure 5.6 illustrates the sensitivity of the two-grid \((1, 2)\)-cycle and the IR iterations to the depth of the boundary relaxation region. The ICG iterations are not sensitive to the presence of boundary relaxation.

![Fig. 5.6. Convergence rates for the Laplace equation with Neumann boundary conditions. Boundary relaxation is applied once before residual restriction.](image)

The test results can be summarized as follows: For small depths of the boundary relaxation region, \( 0 \leq \text{depth} \leq 3 \), the convergence rates of both iterations are not satisfactory. For \( \text{depth} \geq 4 \) the desired convergence rates of 0.12 per cycle are achieved. The performance of iterations is not sensitive to further expansions of the boundary relaxation region. Thus, the value \( \text{depth} = 4 \) is considered optimal. The optimal depth depends on the discretization stencils for boundary and interior equations, but does not depend on the grid size. A smaller optimal boundary relaxation depth can be achieved with a more compact discretization of boundary conditions. For example, a two-point wide discretization of the Neumann condition, derived from joint formulation of the interior five-point Laplacian and the central approximation for the normal derivative at the boundary, leads to the optimal \( \text{depth} = 3 \). The optimal \( \text{depth} = 3 \) can also be achieved for noncompact discretization of the Neumann boundary condition if the boundary relaxation procedure is performed at the end of each relaxation sweep rather than once per cycle. Such an arrangement softens
the residual jump at the interface between the boundary relaxation region and the interior. The latter relaxation-based implementation should not obscure the major role of boundary relaxation as facilitating coarse-grid correction. In practice, we are following an empirical rule to relax simultaneously all the discrete (physical and numerical) boundary conditions as well as all the interior equations that share data with the boundary condition equations.

5.4. Example 4: System of constant-coefficient PDEs. In this section we analyze a two-grid solver for a system of constant-coefficient equations related to the full Newton linearization (3.6) for the plane-flow solution (3.2) with \( a = 1 \). The system of constant-coefficient differential equations is formulated as

\[
\mathbf{L}(\mathbf{q}) \equiv \begin{pmatrix}
\bar{Q}_u - 1 & 0 & \partial_x c

0 & \bar{Q}_v + 1 & \partial_y c

-2\partial_x c & 2\partial_y c & \Delta h
\end{pmatrix} \mathbf{q} = 0,
\]

where \( \mathbf{q} = (u, v, p)^T \), \( \bar{Q}_u = \bar{u}\partial_x u + \bar{v}\partial_y u \), and \( \bar{u} \) and \( \bar{v} \) are positive constants. The computational domain is a rectangle \((x, y) \in [-\bar{u} - X, -\bar{u} + X] \times [\bar{v} - Y, \bar{v} + Y] \) with

\[
0 < X \leq \frac{\bar{u}}{2}, \quad 0 < Y \leq \frac{\bar{v}}{2}.
\]

The latter conditions, referred as compatible-coefficient restrictions, are introduced to ensure that the constant-coefficient problem provides a good relative \( \epsilon \)-accuracy (\( \epsilon = 0.5 \)) in approximating the full-Newton linearization of the plane flow in the vicinity of \( x_0 = (-\bar{u}, \bar{v}) \). A uniform Cartesian grid with a minimum of four mesh intervals in each spatial direction is applied. The physical boundary conditions are the following: \( u, v, \) and \( \partial_x p \) are given at the inflow \( x = -\bar{u} - X \); \( p \) is given at the outflow \( x = -\bar{u} + X \); and periodicity is assumed in the \( y \)-direction \((\mathbf{q}(x, \bar{v} - Y) = \mathbf{q}(x, \bar{v} + Y))\).

The normal derivative of the pressure, \( \partial_x p \), at the inflow boundary is discretized with the second-order accurate, three-point, one-sided stencil. The numerical closure equations include central differencing for the \( x \)-derivative in the operator \( \bar{Q}_u \) at the interior grid points adjacent to the inflow boundary; at the outflow, the \( x \)-derivative of the pressure in the first momentum equation is discretized with the three-point upwind stencil. The exact solution of the problem is \( \mathbf{q} \equiv 0 \).

The relaxation scheme consists of three steps.

(i) Line-implicit Gauss-Seidel relaxation is performed for the Laplace operator in the pressure equation with frozen \( u \) and \( v \). In this relaxation, all the pressure equations defined at a vertical grid line with the same \( y \)-coordinate are relaxed simultaneously; this \( y \)-implicit relaxation scheme proceeds in the upstream direction, from the outflow to the inflow.

(ii) The boundary relaxation (depth = 5) is performed near the inflow boundary. All the equations defined at the five-grid lines adjacent to the inflow boundary (the boundary included) are solved simultaneously. Values of \( u, v, \) and \( p \) in the interior beyond the boundary relaxation region remain unchanged.

(iii) The momentum equations are downstream marched for \( u \) and \( v \) with \( p \) being fixed.

At the end of the relaxation sweep, the residuals of the momentum equations are zero.
The LMF analysis for this solver provides the following estimates. The Fourier symbol of the relaxation scheme is computed as

\[
\hat{S} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} - \begin{pmatrix}
\hat{Q} - 1 & 0 & \hat{\partial}_x \\
0 & \hat{Q} + 1 & \hat{\partial}_y \\
0 & 0 & GS^h
\end{pmatrix}^{-1} \begin{pmatrix}
\hat{Q} - 1 & 0 & \hat{\partial}_x \\
0 & \hat{Q} + 1 & \hat{\partial}_y \\
-2\hat{\partial}_x & 2\hat{\partial}_y & \Delta^h
\end{pmatrix},
\]

(5.9)

where

\[
\hat{Q} = \frac{\bar{u}}{h} \left( \frac{3}{2} - 2e^{-i\theta_x} + \frac{1}{2}e^{-i2\theta_x} \right) + \frac{\bar{v}}{h} \left( \frac{3}{2} - 2e^{-i\theta_y} + \frac{1}{2}e^{-i2\theta_y} \right),
\]

\[
\hat{\partial}_x = \frac{1}{h} \sin(\theta_x),
\]

\[
\hat{\partial}_y = \frac{1}{h} \sin(\theta_y),
\]

\[
\Delta^h = \frac{1}{h^2} \left( 2\cos(\theta_x) + 2\cos(\theta_y) - 4 \right),
\]

\[
GS^h = \frac{1}{h^2} \left( e^{i\theta_x} + 2\cos(\theta_y) - 4 \right),
\]

\[
\theta_x \text{ and } \theta_y \text{ are normalized Fourier frequencies, } |\theta_x| \leq \pi, |\theta_y| \leq \pi.
\]

The smoothing factor, \(\mu\), is

\[
\mu = \max_{\theta^h} \rho(\hat{S}),
\]

(5.11)

where \(\rho(\hat{S})\) is the spectral radius of the matrix \(\hat{S}\), and the maximum is taken over the set of high-frequency Fourier modes \(\theta^h\) defined in (5.3).

The unrestricted LMF analysis considers the infinite domain with no compatible-coefficient restrictions (5.8) applied. This analysis predicts divergence of the relaxation scheme for high-frequency error components. For \(\theta_x = \pi/2, \theta_y = -\pi/2,\) and \(\bar{u} = \bar{v} = 1/2\), the convection symbol \(\hat{Q} = 1\) and the smoothing factor \(\rho(S^h) = \infty\). Imposing the compatible-coefficient restrictions (5.8) together with the assumption that there are at least four mesh intervals in each dimension effectively implies \(\bar{u} \geq 4, \bar{v} \geq 4\). With these restrictions, the LMF analysis predicts the smoothing factor of 0.53 on the \(4 \times 4\) grid and factors better than 0.5 at finer grids converging to \(1/\sqrt{5}\); this latter value is the smoothing factor of line relaxation for the scalar Laplace equation.

<table>
<thead>
<tr>
<th>N</th>
<th>Relaxation</th>
<th>Two-grid</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Smoothing</td>
<td>Amplification</td>
</tr>
<tr>
<td>8</td>
<td>0.51</td>
<td>1.00</td>
</tr>
<tr>
<td>16</td>
<td>0.51</td>
<td>1.14</td>
</tr>
<tr>
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<tr>
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<td>0.50</td>
<td>1.46</td>
</tr>
<tr>
<td>128</td>
<td>0.50</td>
<td>1.80</td>
</tr>
</tbody>
</table>

For \(\bar{u} = \bar{v} = 1\), the relaxation amplification factors are bounded by 1 everywhere except for the characteristic errors. As an example, Table 5.1 collects the relaxation smoothing and amplification factors (the pressure equation is relaxed with lexicographic pointwise Gauss-Seidel relaxation), as well as the amplifications factors of the corresponding two-grid (1, 2)-cycle. As noted previously, the smoothing factors are good. The amplification factors are erratic. On finer grids, the amplification factor
Fig. 5.7. Relaxation amplification factor.

Fig. 5.8. Convergence rates of the pressure equation residuals in solutions of the constant-coefficient problem (5.7). Grid 257 × 33.

tends to infinity for certain smooth (but not the smoothest) characteristic frequencies of the convection operator \((\theta_x = -\theta_y)\), for which the symbol \(\tilde{Q}\) approaches 1. On coarse grids, the smallest possible value of \(|\tilde{Q} - 1|\) is determined by the mesh size \(h\) and the frequency increment \(\delta\theta\) and may vary nonmonotonically; that explains the irregular behavior of the amplification factors. For illustration purposes, the results for a nonstandard 59 × 59 grid are added to the table. At this grid, \(h = 1/58\), and, for \(\theta_x = -\theta_y = \frac{8}{58}\pi\), \(\tilde{Q} = 0.9909\). Figure 5.7 illustrates the relaxation amplification factor over a part of the frequency domain. More details of the analysis can be found in [17]. The LMF analysis does not account for the accuracy propagation from the inflow boundary, that is a major convergence mechanism for this two-grid cycle. Employment of a downstream marching in the relaxation scheme on a domain restricted as (5.8) ensures that the smooth characteristic errors, for which the LMF analysis predicts divergence, are strongly affected by the inflow boundary.

Figure 5.8 shows the convergence rates of the \(L_2\)-norm of the pressure-equation residuals for the four iterative methods solving problem (5.7) with \(\bar{u} = \bar{v} = 1\) on the computational domain \((x, y) \in [-\frac{3}{2}, -\frac{1}{2}] \times [\frac{15}{16}, \frac{17}{16}]\) covered by the uniform isotropic
257 × 33 Cartesian grid. The size and shape of the domain have been chosen to satisfy compatible-coefficient restriction (5.8) and to have an isotropic grid with a large number of grid nodes between the inflow and outflow boundaries. The plot marked "Two-grid (1, 2)-cycle" shows the convergence rates of a two-grid cycle solving (5.7). The cycle includes the three-step relaxation scheme described above, full coarsening, full-weighting residual restriction, and bilinear prolongation. We use the coarse-grid discretization identical to (5.7), therefore, the assignment for relaxation is extended. That is, we suggest that downstream marching of the momentum equations efficiently reduces the smooth characteristic errors that cannot be approximated on the coarse grid. To illustrate the importance of this extended assignment for relaxation, we include two different IR iterations: IR iteration of the first type, IR(p) iteration, replaces step (i) in the relaxation scheme with the averaging procedure (2.5) for the pressure only. Boundary relaxation and downstream marching at steps (ii) and (iii) are performed as in actual relaxation. The second type of IR iterations is IR(uvp) iterations, where steps (i), (ii), and (iii) are replaced by averaging (2.5) for $u$, $v$, and $p$, and no downstream marching is performed.

Convergence rates of the actual cycle are better than convergence rate of the reference cycle with IR(uvp) iterations, indicating the efficiency goal is achieved. ICG iterations and IR(p) iterations provide an accurate estimate for the convergence of the actual cycle. The convergence rates of the ICG iterations and the two-grid cycle are virtually indistinguishable; the IR(p) convergence rates are predictably somewhat worse because line relaxation damps certain intermediate (neither high-frequency nor very smooth) error components significantly better than the ideal averaging procedure (2.5). With progressive grid refinement, the (not shown) convergence rates of the IR(uvp) iterations tend to the value 0.75; this is in agreement with the LMF analysis predictions for convergence rates of full-coarsening multigrid methods applied to a second-order discretization of the convection equation.

Although not shown in the figure, on larger domains (e.g., $[-5, 3] \times [\frac{1}{2}, \frac{3}{2}]$), accuracy propagation from the inflow boundary does not dominate the entire computational domain; divergence of the two-grid cycle is indeed observed in numerical tests, as predicted by the LMF analysis. However, on such domains, the compatible-coefficient restrictions (5.8) are violated; therefore, the formulation (5.7) does not provide a relevant constant-coefficient approximation to the variable-coefficient problem (3.6).

6. Multigrid solver for plane stagnation flow. The basic cycle employed in the multigrid algorithm for solution of the discrete equations (3.5) is an FV($\nu_1, \nu_2$) cycle sketched in Figure 6.1. The FV cycle represents a modification of the well-known multigrid F cycle (see, e.g., [27]). The multigrid solutions have been obtained as follows: Random initial approximation has been assigned at each grid, and a full approximation scheme (FAS) version of FV(1,2) multigrid cycle has been used. The nonlinear equations at the fixed coarsest grid have been solved exactly; thus, finer grids used progressively more levels of multigrid. Compact formulations for discrete boundary conditions at the inflow and tangency boundaries have been implemented (see section 3 and [17]). The relaxation scheme included four steps: (1) point Gauss–Seidel relaxation of the pressure equation performed in a downstream order, (2) downstream relaxation of momentum equations, (3) boundary relaxation ($depth = 3$) at the inflow boundary, and (4) boundary relaxation ($depth = 3$) at the tangency boundary. Within the FAS cycle, the four-step relaxation scheme, full coarsening, full-weighting restriction of residuals, and bilinear prolongation of corrections have been used with direct injection of the nonlinear solution to the coarser level.
Multigrid convergence has been studied for two plane stagnation flow regimes illustrated in Figure 6.2: (1) deep stagnation regime, where stagnation point is a part of the computational boundary; and (2) the regular stagnation flow, where the computational domain is shifted away from stagnation. For efficient solutions of stagnation flows in general curvilinear geometries, we refer readers to [17].

The convergence of the $L_2$-norm of residuals for the regular stagnation regime is shown in the left plot of Figure 6.3. Diamonds correspond to the residuals of the pressure equation; circles and squares denote residuals of the $u$- and $v$-momentum equations, respectively. The largest residuals at each cycle are those of the pressure equation with the momentum residuals generally 1-2 orders of magnitude smaller. The computational results are obtained on the unit-square domain centered at $(x, y) = (-10.5, 10)$. The finest isotropic grid is $65 \times 65$ and the coarsest grid level was $5 \times 5$. Because there is no discretization error for the plane-flow equations, the initial approximations on all grids is taken as random perturbations of the exact solutions for all variables. The convergence rates are grid-independent and very fast, more than an order of magnitude per cycle.
The right plot of Figure 6.3 illustrates the results of the IR and ICG iteration analysis in application to the nonlinear regular stagnation flow equations. Convergence rates of four iterations are compared on the $65 \times 65$ grid: (1) the actual two-grid cycle, (2) the ICG iterations, (3) IR(p) iterations, where explicit full-weighting averaging of algebraic error in the pressure replaces the relaxation of the pressure equation, and (4) IR(uvp) iterations where the entire relaxation sweep is replaced with full-weighting averaging of algebraic errors in $u$, $v$, and $p$. The convergence rates of the IR(uvp) iterations, which are significantly worse than the convergence rates of other iterations, point to some smooth (characteristic) error components that cannot be reduced in the coarse-grid correction. Fast convergence rates in IR(p) iterations confirm that the slow components are efficiently reduced in the downstream marching of the momentum equations. The reference cycle for these nonlinear computations is the same as for constant-coefficient equations in section 5.4, providing the reference convergence rates of 0.18 per cycle. The actual cycle, IR(p) and ICG iterations converge with better asymptotic rates. Close similarity between convergence rates of the actual two-grid cycle, ICG iterations, and IR(p) iterations indicate the optimal efficiency of the actual cycle.

Figure 6.4 presents convergence results for the deep stagnation flow regime. The
convergence rates are grid-independent but show a slight deterioration to approximately 0.14 per cycle asymptotically and a maximal (worst) convergence rate of approximately 0.21. The same four iterations described above have been performed for analysis of convergence in the deep stagnation regime. IR(uvp) iterations again converge slowly; convergence rates of other iterations are faster than rates of the reference cycle and very close to each other, confirming that the developed multigrid algorithm is optimally efficient.

7. Discussion. Some limitations of the classical LMF analysis that make the LMF analysis inapplicable to stagnation-flow problems have been discussed. A concept of $\epsilon$-neighborhood has been introduced to restrict the computational domain to subdomains where there are accurate constant-coefficient approximations to the target variable-coefficient (or nonlinear) problem. With this restriction, some areas of the stagnation flow are inaccessible to the LMF analysis.

A new general approach to quantitative analysis of sophisticated multigrid solvers has been presented. The approach emphasizes the design stage of the process of developing efficient multigrid solutions. Each part of a multigrid solver is assumed to perform a certain task that is well understood and clearly formulated. The new analysis introduces IR and ICG iterations as tools to probe the complementary parts of a multigrid cycle, relaxation and coarse-grid correction. In a practical multigrid solver, one of these parts is replaced with an idealized imitation. The idealized imitation is not an optimally efficient procedure; it is rather a procedure that assuredly fulfills the task assigned to the corresponding part of the multigrid solver. The tests presented in this paper show that comparisons of the convergence rates of IR and ICG iterations with the actual two-grid cycle clearly detect deficiencies of the cycle. The convergence of the actual cycle can be improved either by adjusting parts of the cycle to better perform the assigned tasks or by redesigning the algorithm (redefining some tasks to parts of the cycle) to efficiently reduce slowly converging errors.

The IR and ICG iterations can be viewed as a numerical extension of the LMF analysis to problems where direct analytical methods are unavailable. The concept of isolating and probing parts of an algorithm by implementing idealized imitations for other parts is very general. The major disadvantage of this new analysis (shared by the LMF analysis as well) is that the analysis is not constructive; it detects problems but does not suggest remedies. However, problem detection is a big step forward for complicated applications where no other quantitative analysis methods are available. ICG and IR iterations have already been applied to some complicated nonlinear CFD problems with complex geometries on unstructured grids. The results of these applications have been encouraging in identifying the reasons for relatively slow convergence of practical solvers. In some cases, we have been able to cure the problems, e.g., IR and ICG analysis helped to develop TME solvers for stagnation-flow problems in general geometries [17]; in other cases (unstructured grid applications), we are still in search for adequate remedies.

There are many possible directions to extend the IR and ICG analysis. One can use the spectral radius of the idealized iterations as an estimate for the asymptotic convergence of the actual cycle. Such an extension may add rigor to the analysis, but probably limits its scope to problems of relatively low complexity, where computing the spectral radius is viable. Another possible extension is toward algebraic multigrid (AMG) [4, 10, 21, 24]. A usual assignment for an AMG relaxation is to reduce errors producing large residuals, i.e., the errors corresponding to large eigenvalues of the corresponding matrix; an AMG “coarse-grid” correction is typically
assigned to reduce algebraically smooth errors that are not necessarily geometrically smooth. The assignments extensively rely on the specific matrix; it is hardly possible to design general idealized iterations that would work for any AMG solver. However, assuming restriction, $R_h^H$, and prolongation, $P_h^H$, that preserve algebraically smooth functions, the ICG correction (2.6) is a good choice for general AMG applications. This choice emphasizes that errors that do not change much when transferred to the coarse grid, and then back to the fine grid, are designated as “smooth” to be handled by the coarse-grid operator. It follows that errors that do change much by the intergrid transfers are “nonsmooth” and need to be handled well by the smoother. A good idealized smoother must provide fast convergence of the reference cycle; a possible choice for IR is the composition of restriction and prolongation operators, $P_h^H R_h^H$. With this choice, the eigenvalues, $\lambda_{\text{ref}}$, of the iteration matrix of the reference cycle,

$$
\left( P_h^H R_h^H \right)^{\nu_1} \left( E_h - P_h^H R_h^H \right) \left( P_h^H R_h^H \right)^{\nu_2},
$$

are all in the form

$$
\lambda_{\text{ref}} = \lambda^\nu \left( 1 - \lambda \right),
$$

where $\lambda$ denotes any eigenvalue of the matrix $P_h^H R_h^H$, and $\nu = \nu_1 + \nu_2$ is the total number of relaxations performed per one coarse-grid correction. The expression (7.2) achieves its local maximum when $\lambda = \nu / (\nu + 1)$, yielding

$$
|\lambda_{\text{ref}}| = \frac{\nu^\nu}{(\nu + 1)^{\nu + 1}},
$$

e.g., $\frac{1}{3}$, $\frac{4}{27}$, and $\frac{27}{256}$ for $\nu = 1$, 2, and 3, respectively, which seems very satisfactory.

Of course, $|\lambda_{\text{ref}}|$ might attain its global maximum on the “boundary,” i.e., on the convex hull of the eigenvalues of $P_h^H R_h^H$ in the complex plane. If all eigenvalues of $P_h^H R_h^H$ are real (e.g., when $P_h^H R_h^H$ is symmetric), to find the global maximum, one only needs to consider the largest and smallest values of $\lambda$. In the particular common case where $R_h^H$ is the (suitably scaled) transpose of $P_h^H$, $P_h^H R_h^H$ is symmetric, positive semidefinite, so all of its eigenvalues are real and nonnegative. Furthermore, if $P_h^H$ and $R_h^H$ are properly scaled, the largest value of $\lambda$ does not exceed 1, i.e., no eigenfunction is amplified by restriction followed by prolongation. Therefore, the spectral radius of the reference cycle is indeed given by (7.3). This will generally be the case, when the eigenvalues of $P_h^H R_h^H$ are real, except in unrealistic cases where $P_h^H R_h^H$ amplifies some eigenfunction significantly or changes signs of its elements. If $P_h^H R_h^H$ has complex eigenvalues, there may be some (often slight) deterioration of convergence of the reference cycle (which can be analyzed in a similar way). In this case, performing IR iterations which alternate between $(P_h^H R_h^H)^T$ and $P_h^H R_h^H$ may yield improved results.

REFERENCES


