First-Order Hyperbolic System Method

‘If you have a CFD book for hyperbolic problems, you have a CFD book for all problems.’

Hiro Nishikawa
National Institute of Aerospace
History of First-Order Hyperbolic System Method

2004  *First-order* system employed for high-order RD method (*steady*)

\[
\begin{align*}
    u_t &= \nu u_{xx} \quad \rightarrow \quad & u_t &= \nu p_x \\
    p &= u_x
\end{align*}
\]

Mixed formulation

2005  34th VKI CFD Lecture Series: Very High Order Schemes
2006  Time-derivative added, and the system becomes *hyperbolic.*

\[
\begin{align*}
    u_t &= \nu p_x \\
    p_t &= \frac{(u_x - p)}{T}\quad \text{Cattaneo(1958)}
\end{align*}
\]

2007  JCP Part I  -  **First-Order Hyperbolic System Method**

2009  AIAA 2009
2010  JCP Part II  -  **Unification of Advection and Diffusion**
2011  AIAA 2011  -  **Hyperbolic Navier-Stokes System**
Hyperbolic Model for Diffusion

Diffusion Equation
\[ u_t = \nu u_{xx} \]

Hyperbolic Model for Diffusion
\[ \begin{align*}
    u_t &= \nu p_x \\
    p_t &= (u_x - p)/T_r
\end{align*} \]

Eigenvalues are real: \( \lambda = \pm \sqrt{\frac{\nu}{T_r}} \)

There have been attempts to solve the hyperbolic model instead of diffusion, targeting applications to high-order moment equations (rarefied gas).

Jin and Levermore (1996)
Lowrie and Morel (2002)

Advantages: compact, parallel, physics, etc.
Difficulty is the stiffness due to extremely small Tr.

From PhD thesis of M. Arora, University of Michigan, 1995
First-Order Hyperbolic System Method

Basic Idea (JCP 2007):

Hyperbolic model becomes the diffusion equation in the steady state for any relaxation time, $Tr$.

\[
\begin{align*}
0 &= \nu p_x \\
0 &= (u_x - p)/Tr
\end{align*}
\]

\[
0 = \nu p_x \quad \rightarrow \quad p = u_x \quad \rightarrow \quad 0 = \nu u_{xx}
\]

We can compute steady solution to diffusion by integrating the hyperbolic system in time.

Tr is a free parameter - Stiffness is not an issue for steady computations. Advection scheme for diffusion - Goodbye to diffusion schemes...
Classical Pseudo-Transient Method

Solution to the steady equation,
\[ 0 = \nu (u_{xx} + u_{yy}) \]
can be computed by integrating a pseudo-transient equation,
\[ u_\tau = \nu (u_{xx} + u_{yy}) \]
towards the steady state.

The pseudo-transient equation can be *anything* as long as it is stable and reduces to the original equation in the steady state.

For example, a wave equation (hyperbolic):
\[ u_{\tau\tau} = \nu (u_{xx} + u_{yy}) \]

Or the first-order hyperbolic system:
\[ u_\tau = \nu (p_x + q_y), \]
\[ p_\tau = (u_x - p)/T_r, \]
\[ q_\tau = (u_y - q)/T_r. \]

Note: It is elliptic in space.
Two Different Directions

- Moment Equations - rarefied gas
- Navier-Stokes Equations
  High-order PDEs

Stiff Relaxation
Jin, Roe, Van Leer, others

Tr → 0

First-Order Hyperbolic
Nishikawa

Arbitrary Tr

\[ u_t = \nu p_x \]
\[ p_t = (u_x - p)/Tr \]

Target equations and applications are different.
Relaxation Time

Tr can be defined to accelerate convergence towards steady state.

E.g., require that the relaxation speed be comparable to the characteristic wave speed, possibly to enhance the convergence:

\[
\frac{L_r}{T_r} = \lambda \left( = \sqrt{\frac{\nu}{T_r}} \right) \rightarrow T_r = \frac{L_r^2}{\nu}, \quad L_r = O(1).
\]

This is significantly larger than that in stiff relaxation methods:

\[
T_r \propto \nu
\]

for small diffusion coefficient.

Lr is a constant that can be just 1.0 or can be determined to optimize properties of schemes (JCP2007) or optimize the condition number of the differential system (JCP2010).

In effect, we are adjusting the relaxation time so as to keep the system strongly hyperbolic towards the steady state.
Once reach the steady state, we obtain solution to diffusion, and the diffusive flux, \( p \), to the same accuracy.

Eigenvalues are real: \( \lambda = \pm \sqrt{\frac{\nu}{T_r}} = \pm \frac{\nu}{L_r} \)

Waves travelling to the left and right at the \textit{same finite} speed.

E.g., Upwind scheme for diffusion:

\[
F_{j+1/2} = \frac{1}{2} \left[ F_{j+1} + F_{j} \right] - \frac{1}{2} |A| \left( U_{j+1} - U_{j} \right)
\]

Upwinding results in a symmetric stencil due to the symmetric wave structure.

CFL condition:

\[
\Delta t \leq \frac{h}{\max \text{ wave speed}} = \frac{h}{\nu/L_r} = O(h)
\]
Why Same Order of Accuracy?

If we discretize a hyperbolic system,

$$U_t + AU_x = Q$$

By, say, 2nd-order finite-volume method, the solution will be 2nd-order accurate for all variables in $U$: $u$ and $p$.

Integrated hyperbolic-system formulation ensures equal order of accuracy of all variables in the solution vector.
O(h) Time Step

Typical Scheme (e.g., Galerkin): \[ u_{j}^{n+1} = u_{j}^{n} + \nu \Delta t \frac{u_{j+1}^{n} - 2u_{j}^{n} + u_{j-1}^{n}}{h^2} \]

\[ \Delta t \leq \frac{h^2}{2\nu} = O(h^2) \rightarrow n_{\text{steady}} \propto \frac{1}{\Delta t} = O(1/h^2) \]

Hyperbolic scheme - CFL Condition:

\[ \Delta t \leq \frac{h}{\nu/L_r} = O(h) \rightarrow n_{\text{steady}} \propto \frac{1}{\Delta t} = O(1/h) \]

As fast as the optimal SOR.

O(1/h) speed-up to reach the steady state (1/2/3D).

\[ O(1/h) = O(N^{1/D}) : O(100) \text{ times faster for 1 million nodes.} \]

\[ O(1000) \text{ times faster for 1 billion nodes.} \]
1. **Hyperbolic Methods for Diffusion**
   Riemann solvers, multi-D upwind, high-order, non-oscillatory, etc.

2. **O(h) Time Step for Diffusion**
   Rapid convergence to a steady state by explicit schemes.

3. **Accurate Diffusive Fluxes (Derivatives)**
   Same order of accuracy as the main variables.
   Boundary conditions made simple (**Neumann** -> **Dirichlet**).

*If you have a CFD book for hyperbolic systems,*

*you have a CFD book for diffusion.*
Demonstration

1D code is available at http://www.cfdbooks.com
Hyperbolic Advection-Diffusion System

\[ u_t + a \ u_x = \nu \ u_{xx} \quad \rightarrow \quad u_t + a u_x = \nu \ p_x \]

\[ p_t = \frac{(u_x - p)}{T_r} \quad a > 0 \]

Vector Form: \( U_t + AU_x = S \quad U = \begin{bmatrix} u \\ p \end{bmatrix}, \quad A = \begin{bmatrix} a & -\nu \\ -1/T_r & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ -p/T_r \end{bmatrix} \)

\( T_r \) is derived by requiring \( \frac{L_r}{T_r} = \lambda \quad \rightarrow \quad T_r = \frac{L_r}{a + \nu/L_r} \)

Eigenvalues: \( \lambda = -\frac{\nu}{L_r}, \quad a + \frac{\nu}{L_r} \)

\( L_r \) is derived by optimizing the condition number of the system:

\[ L_r = \frac{1}{2\pi} \left[ \frac{Re_{\pi}}{\sqrt{1 + Re_{\pi}^2 + 1}} + \frac{2}{\sqrt{1 + Re_{\pi}^2 + 1}} \right], \quad Re_{\pi} = \frac{a(1/\pi)}{\nu} \]

Unification of advection and diffusion as a single hyperbolic system
Hyperbolic scheme - CFL Condition:

\[ \Delta t \leq \frac{h_{\text{min}}}{a + \nu/L_r} = O(h) \]

Typical Scheme (e.g., Galerkin):

\[ u_{j}^{n+1} = u_{j}^{n} + \frac{2\Delta t}{x_{j+1} - x_{j-1}} \left[ -a \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2} + \nu \left( \frac{u_{j+1}^{n} - u_{j}^{n}}{x_{j+1} - x_{j}} - \frac{u_{j}^{n} - u_{j-1}^{n}}{x_{j} - x_{j-1}} \right) \right] \]

\[ \Delta t \leq \frac{h_{\text{min}}}{a + \nu/h_{\text{min}}} = O(h^2) \]

O(1/h) speed-up through the diffusion limit.
For Any Discretization

1. **One Scheme for Advection-Diffusion System**
   Single hyperbolic scheme for the whole advection-diffusion system.
   No need to combine two different schemes (advection and diffusion).

2. **Uniform Accuracy for All Reynolds Numbers**
   No blending function necessary in RD and continuous Galerkin methods.

3. **O(h) Time Step for All Reynolds Numbers**
   Rapid convergence to a steady state by explicit schemes.

4. **Accurate Diffusive Fluxes (Derivatives)**
   Same order of accuracy as the main variables.
   Boundary conditions made simple (Neumann -> Dirichlet).

5. **Hyperbolic Methods Applicable to Adv-Diff**
   Riemann solvers, limiters, Multi-D Upwind, high-order, etc.
where \( u(0)=0 \) and \( u(1)=1 \), and

\[
q(x) = \frac{\pi}{Re} \left[ a \cos(\pi x) + \pi \nu \sin(\pi x) \right], \quad Re = a/\nu
\]

**Upwind scheme constructed for hyperbolic advection-diffusion system. JCP2010**

Stretched Grids: 33, 65, 129, 257 points.

CFL = 0.99, Forward Euler time-stepping.

Residual reduction by 5 orders of magnitude

p is NOT given but computed on the boundary.
1D Convergence Results

The number of iterations (time steps) to reach the steady state:

<table>
<thead>
<tr>
<th>log_{10} Re</th>
<th>Number of Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>33 nodes</td>
</tr>
<tr>
<td>−3.0</td>
<td>2976</td>
</tr>
<tr>
<td>−2.0</td>
<td>2979</td>
</tr>
<tr>
<td>−1.5</td>
<td>2986</td>
</tr>
<tr>
<td>−1.0</td>
<td>3010</td>
</tr>
<tr>
<td>−0.5</td>
<td>3086</td>
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<tr>
<td>0.0</td>
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<tr>
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<td>3175</td>
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<tr>
<td>1.0</td>
<td>3999</td>
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<tr>
<td>1.5</td>
<td>3062</td>
</tr>
<tr>
<td>2.0</td>
<td>3214</td>
</tr>
<tr>
<td>3.0</td>
<td>3286</td>
</tr>
</tbody>
</table>

Nearly independent of the Reynolds number.

\[
L_r = \frac{1}{2\pi} \left[ \frac{Re_{\pi}}{\sqrt{1 + Re_{\pi}^2} + 1} + \sqrt{1 + \frac{2}{\sqrt{1 + Re_{\pi}^2} + 1}} \right], \quad Re_{\pi} \equiv \frac{a(1/\pi)}{\nu}
\]
Comparison with the Galerkin Scheme

The speed-up factor, $O(1/h)$, grows for finer mesh
1D Accuracy Results

Maximum errors (L_{inf}): 

2^{\text{nd}}-\text{Order accurate for both } u \text{ and } p \text{ for all Reynolds numbers.}
Hyperbolic Navier-Stokes System

~ A Farewell to Traditional Navier-Stokes Codes ~
First-Order Navier-Stokes System

Navier-Stokes Equations
Of course, not hyperbolic...

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p - \tau)}{\partial x} = 0, \\
\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u H - \tau u + q)}{\partial x} = 0,
\]

\[
\tau = \mu_v \frac{\partial u}{\partial x}, \\
q = -\frac{\mu_h}{\gamma(\gamma - 1)} \frac{\partial T}{\partial x},
\]

\[
\left(\mu_v = \frac{4}{3} \mu, \quad \mu_h = \frac{\gamma \mu}{Pr}\right)
\]

Two systems are equivalent in the steady state.

First-Order Navier-Stokes System
Nishikawa, AIAA 2011-3043

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p - \tau)}{\partial x} = 0, \\
\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u H - \tau u + q)}{\partial x} = 0,
\]

\[
\frac{\partial \tau}{\partial t} - \frac{1}{T_v} \left( \mu_v \frac{\partial u}{\partial x} - \tau \right) = 0, \\
\frac{\partial q}{\partial t} - \frac{1}{T_h} \left( -\frac{\mu_h}{\gamma(\gamma - 1)} \frac{\partial T}{\partial x} - q \right) = 0,
\]

\[
T_v = L^2/\nu_v, \quad T_h = L^2/\nu_h
\]
The inviscid and viscous Jacobians:
\[ PA = PA^i + PA^v, \quad A = \frac{\partial F}{\partial U}. \]
Viscous Term is a Hyperbolic System

Viscous Jacobian has real eigenvalues:

\[ \lambda^v = \pm \sqrt{\frac{\nu_v}{T_v}}, \quad \pm \sqrt{\frac{\nu_h}{T_h}}, \quad 0 \]

Viscous and heating waves

First-Order NS = Inviscid term + Viscous term

Methods for hyperbolic systems are all we need.
Hyperbolic Navier-Stokes System

\[ P^{-1} \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S \]

Intrinsic Features:

1. Methods for hyperbolic systems apply to the whole system.
2. \( O(1/h) \) faster than ‘traditional’ Navier-Stokes codes.
3. Accurate viscous/heat fluxes, of course.

If you have a good method for the Euler, you have a very good method for the Navier-Stokes.

High-order, FVS, Riemann solver, positive or non-oscillatory, implicit, multigrid....
Hyperbolic Navier-Stokes Scheme

Finite-Volume Method:

\[ P_j^{-1} \frac{dU_j}{dt} = -\frac{1}{\Delta x} \left[ F_{j+1/2} - F_{j-1/2} \right] + S_j \]

Upwind Navier-Stokes Flux:

\[ F_{j+1/2} = \frac{1}{2} [F_R + F_L] - \frac{1}{2} P^{-1} |PA| (U_R - U_L) \]

where

\[ |PA| \approx |PA^i| + |PA^v| \]

That’s it. You can now start writing a Navier-Stokes code.
Hyperbolic Navier-Stokes Scheme (Nishikawa, AIAA 2011-3043):

$$\Delta t = \frac{CFL}{\max(|u| + a + a_h)} h = O(h)$$

Traditional Navier-Stokes Scheme (Nishikawa, AIAA 2011-3044):

$$\Delta t = \frac{CFL}{\max(|u| + a + O(1/h))} h = O(h^2)$$

$O(1/h)$ speed-up to reach the steady state (1/2/3D).
Viscous Shock Structure Problem

\[ M_\infty = 3.5, \quad Pr = \frac{3}{4}, \quad \gamma = 1.4, \quad Re_\infty = 25, \quad T_\infty = 400[\text{K}] . \]

Program can be downloaded at http://www.cfdbooks.com
2D Numerical Results

Viscous shock structure problem in 2D domain

Node-centered edge-based FV method

1. *Hyperbolic* scheme (Upwind Navier-Stokes flux) \(9\) equations
2. *Traditional* scheme (Roe flux, Ave-LSQ flux) \(4\) equations

Steady state is reached at residual reduction by 6 orders.

Irregular triangular grids: 533, 867, 1281, 1751, 2349, nodes.
Convergence Results

Hyperbolic scheme is $O(1/h)$ faster in CPU time.
Cost of solving extra equations is overwhelmed by $O(1/h)$ speed-up.
Error Convergence in Main Variables

(a) $L_1$ error of $\rho$.
(b) $L_1$ error of $u$.
(c) $L_1$ error of $v$.
(d) $L_1$ error of $p$. 

Traditional
Hyperbolic
Slope 2
Error Convergence in Viscous/Heat Fluxes

1. Hyperbolic Scheme:
   Simultaneously computed.

2. Traditional Scheme:
   LSQ reconstruction

(a) $L_1$ error of $\tau_{xx}$.
(b) $L_1$ error of $\tau_{xy}$.
(c) $L_1$ error of $q_x$.
(d) $L_1$ error of $q_y$. 
First-Order Hyperbolic System Method

Construct a hyperbolic system for target PDEs, and integrate it towards the steady state.

Demonstrated for Diffusion, Advection-Diffusion, and NS Equations:
- Upwind scheme for viscous term
- $O(1/h)$ speed-up by explicit scheme
- Accurate viscous/heat fluxes on irregular grids

Accurate viscous force and heating rate prediction on unstructured grids come with orders-of-magnitude improvement in efficiency.

The door is now open for applications to various nonlinear systems.
Lots of Lots of Future Work

Non-Oscillatory Schemes – High-Reynolds, shock waves.

Implicit Schemes – Simpler construction, fast iterative linear solvers

Time-Accurate Schemes – Dual-time stepping with fast inner iterations.

Other Discretization Methods – FV, FE, CESE, RD, DG, SV, SD,…..

Other Hyperbolic NS Systems – Include vorticity or solution gradients.

Turbulence Models – All gradients in source terms turned into scalars.

Other Governing Eqns – Incomp. NS, Turbulence, Resistive MHD...

Higher-Order PDEs – Construct a hyperbolic system and integrate it.

And so on and on and on...
Hyperbolic Viscous System is a Coin.

This presentation showed only a portion of a side of the coin (red square).

On the other side of the coin lie a variety of traditional viscous schemes — 2\textsuperscript{nd}-derivatives: The hyperbolic viscous system generates also robust and accurate traditional viscous schemes. That's another story about challenging the common wisdom on viscous discretization.

Why is it a dime?