NIA CFD Seminar, October 4, 2011
Hyperbolic Seminar, NASA Langley, October 17, 2011

First-Order Hyperbolic System Method

'If you have a CFD book for hyperbolic problems,
you have a CFD book for all problems.'

Hiro Nishikawa
National Institute of Aerospace

NIA
NATIONAL INSTITUTE OF AEROSPACE

History of First-Order Hyperbolic System Method

2004 *First-order* system employed for high-order RD method (*steady*)

$$u_t = \nu\, u_{xx} \quad \longrightarrow \quad \begin{aligned} u_t &= \nu\, p_x \\ p &= u_x \end{aligned}$$

Mixed formulation

2005 34th VKI CFD Lecture Series: Very High Order Schemes

2006 Time-derivative added, and the system becomes *hyperbolic.*

$$\begin{aligned} u_t &= \nu\, p_x \\ p_t &= (u_x - p)/T_r \end{aligned}$$

Cattaneo(1958)

2007 JCP Part I - **First-Order Hyperbolic System Method**

2009 AIAA 2009

2010 JCP Part II - **Unification of Advection and Diffusion**

2011 AIAA 2011 - **Hyperbolic Navier-Stokes System**

Hyperbolic Model for Diffusion

Cattaneo(1958)

Diffusion Equation

$$u_t = \nu\, u_{xx}$$

$$t \gg T_r$$

$$\Longleftrightarrow$$

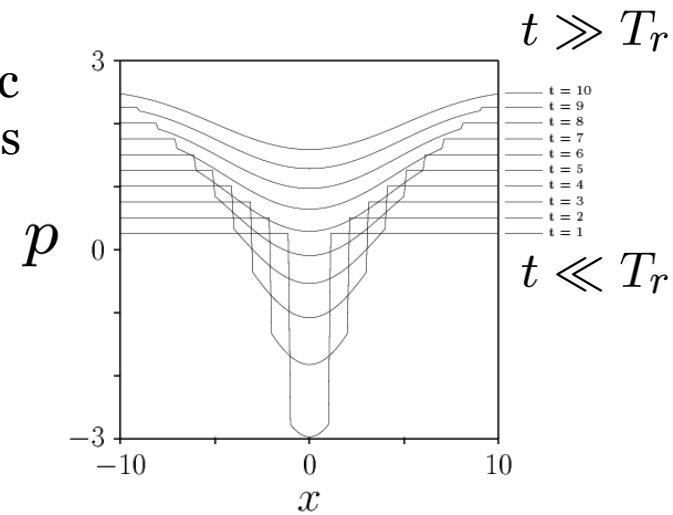
Hyperbolic Model for Diffusion

$$u_t = \nu\, p_x$$
$$p_t = (u_x - p)/T_r$$

Eigenvalues are real : $\lambda = \pm\sqrt{\dfrac{\nu}{T_r}}$

There have been attempts to solve the hyperbolic model instead of diffusion, targeting applications to high-order moment equations (rarefied gas).

Roe, Arora, Hittinger (1993, 2000)
Van Leer, Brown, Suzuki, Khieu(1996, 2008, 2011)
Jin and Levermore (1996)
Lowrie and Morel (2002)



From PhD thesis of M. Arora, University of Michigan, 1995

Advantages: compact, parallel, physics, etc.
Difficulty is the stiffness due to extremely small Tr.

First-Order Hyperbolic System Method

Basic Idea (JCP 2007):

Hyperbolic model becomes the diffusion equation in the steady state for any relaxation time, Tr.

$$\begin{aligned} 0 &= \nu\, p_x \\ 0 &= (u_x - p)/T_r \end{aligned} \quad \rightarrow \quad \begin{aligned} 0 &= \nu\, p_x \\ p &= u_x \end{aligned} \quad \rightarrow \quad 0 = \nu\, u_{xx}$$

We can compute steady solution to diffusion by integrating the hyperbolic system in time.

Tr is a free parameter - Stiffness is not an issue for steady computations.
Advection scheme for diffusion - Goodbye to diffusion schemes...

Classical Pseudo-Transient Method

Solution to the steady equation,

$$0 = \nu \left(u_{xx} + u_{yy} \right)$$

can be computed by integrating a pseudo-transient equation,

$$u_\tau = \nu \left(u_{xx} + u_{yy} \right)$$

towards the steady state.

The pseudo-transient equation can be *anything* as long as it is stable and reduces to the original equation in the steady state.

For example, a wave equation (hyperbolic): $u_{\tau\tau} = \nu \left(u_{xx} + u_{yy} \right)$

Or the first-order hyperbolic system:

$$\begin{aligned} u_\tau &= \nu \left(p_x + q_y \right), \\ p_\tau &= \left(u_x - p \right) / T_r, \\ q_\tau &= \left(u_y - q \right) / T_r. \end{aligned}$$

Note: It is elliptic in space.

Two Different Directions

Moment Equations - rarefied gas

Navier-Stokes Equations
High-order PDEs

Solve all as hyperbolic.

Stiff Relaxation
Jin, Roe, Van Leer, others

First-Order Hyperbolic
Nishikawa

$T_r \to 0$

Arbitrary T_r

$$\begin{aligned} u_t &= \nu\, p_x \\ p_t &= (u_x - p)/T_r \end{aligned}$$

Target equations and applications are different.

Relaxation Time

Tr can be defined to accelerate convergence towards steady state.

E.g., require that the relaxation speed be comparable to the characteristic wave speed, possibly to enhance the convergence:

$$\frac{L_r}{T_r} = \lambda \left(= \sqrt{\frac{\nu}{T_r}} \right) \quad \rightarrow \quad \boxed{T_r = \frac{L_r^2}{\nu}} \, L_r = O(1).$$

This is significantly larger than that in stiff relaxation methods:

$$T_r \propto \nu$$

for small diffusion coefficient.

Lr is a constant that can be just 1.0 or can be determined to optimize properties of schemes (JCP2007) or optimize the condition number of the differential system (JCP2010).

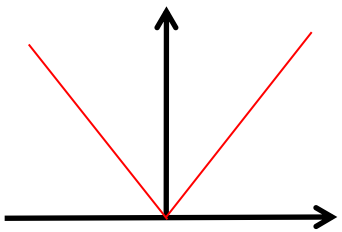
In effect, we are adjusting the relaxation time so as to keep the system strongly hyperbolic towards the steady state.

Upwind Diffusion Scheme

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{Q}, \quad \mathbf{U} = \begin{bmatrix} u \\ p \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -\nu \\ -1/T_r & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0 \\ -p/T_r \end{bmatrix}$$

Eigenvalues are real: $\lambda = \pm\sqrt{\dfrac{\nu}{T_r}} = \pm\dfrac{\nu}{L_r}$

Waves travelling to the left and right at the ***same finite*** speed.



E.g., Upwind scheme for diffusion:

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left[\mathbf{F}_{j+1} + \mathbf{F}_j\right] - \frac{1}{2}|\mathbf{A}|\left(\mathbf{U}_{j+1} - \mathbf{U}_j\right)$$

Upwinding results in a symmetric stencil due to the symmetric wave structure.

CFL condition: $\Delta t \leq \dfrac{h}{\text{max wave speed}} = \dfrac{h}{\nu/L_r} = O(h)$

Once reach the steady state, we obtain solution to diffusion, and the diffusive flux, p, to the same accuracy.

Why Same Order of Accuracy?

If we discretize a hyperbolic system,

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{Q}$$

By, say, 2nd-order finite-volume method, the solution will be 2nd-order accurate for all variables in **U**: u and p.

Integrated hyperbolic-system formulation ensures equal order of accuracy of all variables in the solution vector.

O(h) Time Step

Typical Scheme (e.g., Galerkin): $u_j^{n+1} = u_j^n + \nu\,\Delta t\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}$

$$\Delta t \leq \frac{h^2}{2\nu} = \boxed{O(h^2)} \longrightarrow n_{steady} \propto \frac{1}{\Delta t} = O(1/h^2)$$

Hyperbolic scheme - CFL Condition:

$$\Delta t \leq \frac{h}{\nu/L_r} = \boxed{O(h)} \longrightarrow n_{steady} \propto \frac{1}{\Delta t} = O(1/h)$$

As fast as the optimal SOR.

O(1/h) speed-up to reach the steady state (1/2/3D).

$O(1/h) = O(N^{1/D})$: $O(100)$ times faster for 1 million nodes.

$O(1000)$ times faster for 1 billion nodes.

Whatever the Discretization

1. **Hyperbolic Methods for Diffusion**
   Riemann solvers, multi-D upwind, high-order, non-oscillatory, etc.
2. **O(h) Time Step for Diffusion**
   Rapid convergence to a steady state by explicit schemes.
3. **Accurate Diffusive Fluxes (Derivatives)**
   Same order of accuracy as the main variables.
   Boundary conditions made simple (Neumann -> Dirichlet).

If you have a CFD book for hyperbolic systems,
you have a CFD book for diffusion.

Demonstration

1D code is available at http://www.cfdbooks.com

Nishikawa, JCP 2010

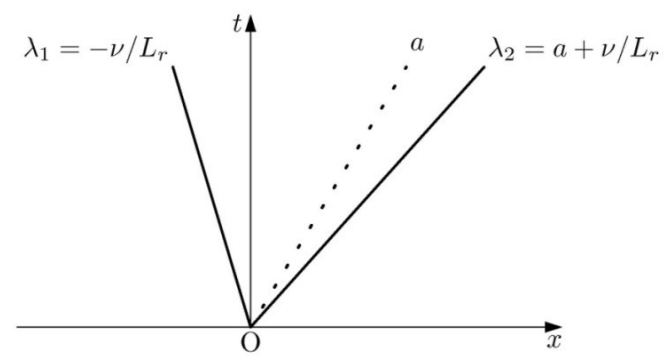
Hyperbolic Advection-Diffusion System

$$u_t + a\,u_x = \nu\,u_{xx} \quad \longrightarrow \quad \begin{aligned} u_t + a u_x &= \nu\,p_x \\ p_t &= (u_x - p)/T_r \end{aligned} \qquad a > 0$$

Vector Form: $\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{S}$ $\quad \mathbf{U} = \begin{bmatrix} u \\ p \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a & -\nu \\ -1/T_r & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ -p/T_r \end{bmatrix}$

Tr is derived by requiring $\dfrac{L_r}{T_r} = \lambda \quad \longrightarrow \quad T_r = \dfrac{L_r}{a + \nu/L_r}$

Eigenvalues: $\lambda = -\dfrac{\nu}{L_r}, \quad a + \dfrac{\nu}{L_r}$



Lr is derived by optimizing the condition number of the system:

$$L_r = \frac{1}{2\pi}\left[\frac{Re_\pi}{\sqrt{1+Re_\pi^2}+1} + \sqrt{1 + \frac{2}{\sqrt{1+Re_\pi^2}+1}}\right], \quad Re_\pi \equiv \frac{a(1/\pi)}{\nu}$$

Unification of advection and diffusion as a single hyperbolic system

Nishikawa, JCP 2010

O(h) Time Step

Typical Scheme (e.g., Galerkin):

$$u_j^{n+1} = u_j^n + \frac{2\Delta t}{x_{j+1} - x_{j-1}} \left[-a \frac{u_{j+1}^n - u_{j-1}^n}{2} + \nu \left(\frac{u_{j+1}^n - u_j^n}{x_{j+1} - x_j} - \frac{u_j^n - u_{j-1}^n}{x_j - x_{j-1}} \right) \right]$$

$$\Delta t \leq \frac{h_{min}}{a + \nu / h_{min}} = \boxed{O(h^2)}$$

Hyperbolic scheme - CFL Condition:

$$\Delta t \leq \frac{h_{min}}{a + \nu / L_r} = \boxed{O(h)}$$

O(1/h) speed-up through the diffusion limit.

Nishikawa, JCP 2010

For Any Discretization

1. **One Scheme for Advection-Diffusion System**
   Single hyperbolic scheme for the whole advection-diffusion system.
   No need to combine two different schemes (advection and diffusion).

2. **Uniform Accuracy for All Reynolds Numbers**
   No blending function necessary in RD and continuous Galerkin methods.

3. **O(h) Time Step for All Reynolds Numbers**
   Rapid convergence to a steady state by explicit schemes.

4. **Accurate Diffusive Fluxes (Derivatives)**
   Same order of accuracy as the main variables.
   Boundary conditions made simple (Neumann -> Dirichlet).

5. **Hyperbolic Methods Applicable to Adv-Diff**
   Riemann solvers, limiters, Multi-D Upwind, high-order, etc.

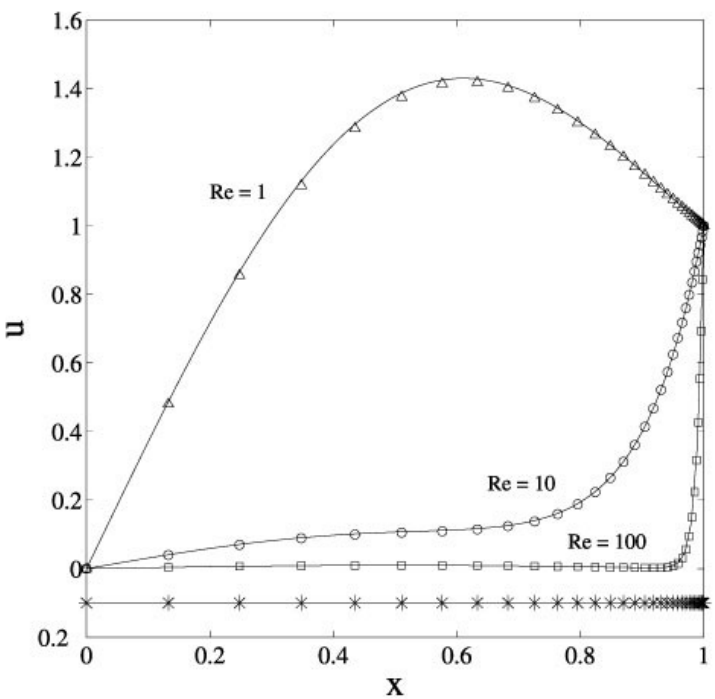
Nishikawa, JCP 2010

1D Test Problem

$$u_t + a u_x = \nu\, u_{xx} + q(x) \quad \text{in } \Omega = (0,1)$$

where u(0)=0 and u(1)=1, and

$$q(x) = \frac{\pi}{Re}\left[a\cos(\pi x) + \pi\nu \sin(\pi x)\right], \qquad Re = a/\nu$$



Upwind scheme constructed for hyperbolic advection-diffusion system. JCP2010

Stretched Grids: 33, 65, 129, 257 points.

CFL = 0.99, Forward Euler time-stepping.

Residual reduction by 5 orders of magnitude

p is NOT given but computed on the boundary.

Nishikawa, JCP 2010

1D Convergence Results

The number of iterations (time steps) to reach the steady state:

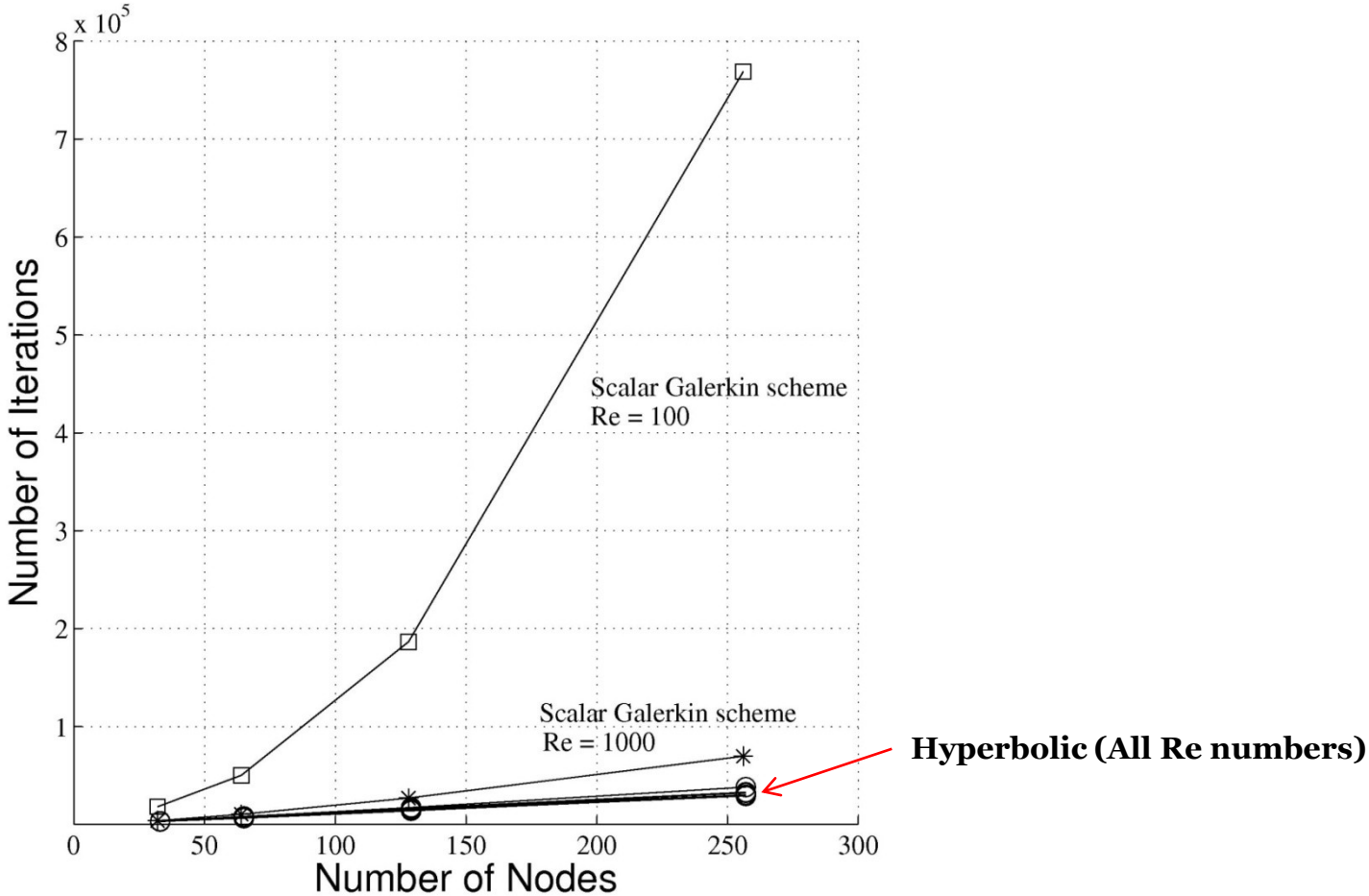
$\log_{10} Re$	Number of Iterations			
	33 nodes	65 nodes	129 nodes	257 nodes
−3.0	2976	7368	14685	29170
−2.0	2979	7376	14700	29199
−1.5	2986	7393	14735	29270
−1.0	3010	7449	14847	29497
−0.5	3086	7629	15218	30244
0.0	3349	8186	16491	32869
0.5	3175	7926	17277	38081
1.0	3999	7735	15428	35747
1.5	3062	7180	15389	32277
2.0	3214	6458	13962	29518
3.0	3286	6877	14355	29893

Nearly independent of the Reynolds number.

$$L_r = \frac{1}{2\pi}\left[\frac{Re_\pi}{\sqrt{1+Re_\pi^2}+1} + \sqrt{1+\frac{2}{\sqrt{1+Re_\pi^2}+1}}\right], \quad Re_\pi \equiv \frac{a(1/\pi)}{\nu}$$

Nishikawa, JCP 2010

Comparison with the Galerkin Scheme

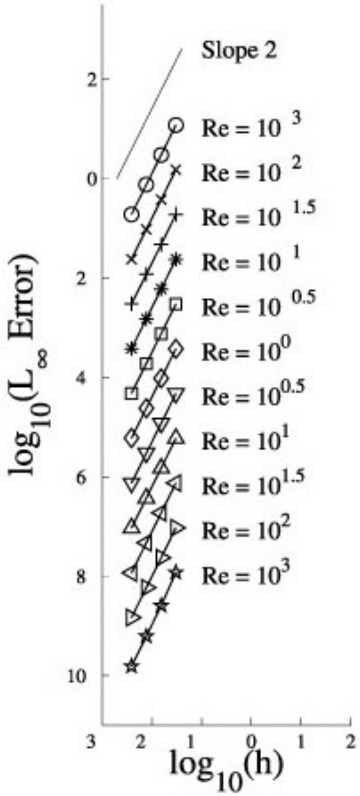


The speed-up factor, O(1/h), grows for finer mesh

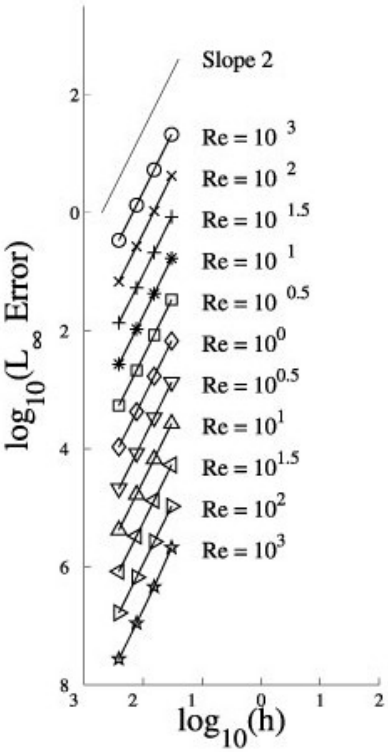
Nishikawa, JCP 2010

1D Accuracy Results

Maximum errors (Linf):



Error of the solution, u



Error of the diffusive flux, p

2nd-Order accurate for both u and p for all Reynolds numbers.

Hyperbolic Navier-Stokes System

~ A Farewell to Traditional Navier-Stokes Codes ~

First-Order Navier-Stokes System

Navier-Stokes Equations

Of course, not hyperbolic...

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} &= 0, \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p - \tau)}{\partial x} &= 0, \\
\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u H - \tau u + q)}{\partial x} &= 0, \\
\tau &= \mu_v \frac{\partial u}{\partial x}, \\
q &= -\frac{\mu_h}{\gamma(\gamma - 1)} \frac{\partial T}{\partial x}.
\end{aligned}
$$

$$\left(\mu_v = \frac{4}{3}\mu, \quad \mu_h = \frac{\gamma \mu}{Pr}.\right)$$

First-Order Navier-Stokes System

Nishikawa, AIAA 2011-3043

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} &= 0, \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p - \tau)}{\partial x} &= 0, \\
\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u H - \tau u + q)}{\partial x} &= 0, \\
\frac{\partial \tau}{\partial t} - \frac{1}{T_v}\left(\mu_v \frac{\partial u}{\partial x} - \tau\right) &= 0, \\
\frac{\partial q}{\partial t} - \frac{1}{T_h}\left(-\frac{\mu_h}{\gamma(\gamma - 1)} \frac{\partial T}{\partial x} - q\right) &= 0,
\end{aligned}
$$

$$T_v = L^2/\nu_v, \quad T_h = L^2/\nu_h$$

Two systems are equivalent in the steady state.

Preconditioned Conservative System

$$\mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & T_v/\mu_v & 0 \\ 0 & 0 & 0 & 0 & T_h/\mu_h \end{bmatrix}, \mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \\ \tau \\ q \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p - \tau \\ \rho u H - \tau u + q \\ -u \\ \dfrac{a^2}{\gamma(\gamma-1)} \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau/\mu_v \\ q/\mu_h \end{bmatrix}$$

Discretization method of this type of system is well known: Local-preconditioning technique.

The inviscid and viscous Jacobians:

$$\mathbf{PA} = \underbrace{\mathbf{PA^i}}_{\text{Euler}} + \underbrace{\mathbf{PA^v}}_{\text{Viscous}}, \quad \mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}.$$

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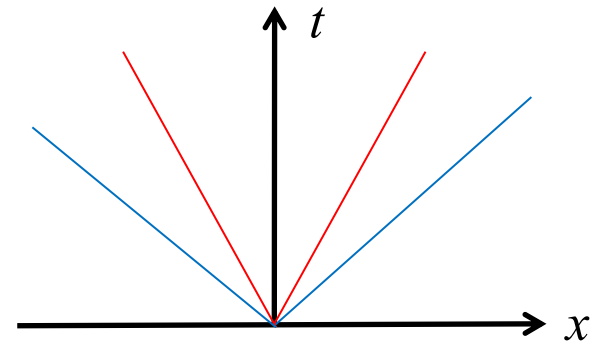
Viscous Term is a Hyperbolic System

Viscous Jacobian has real eigenvalues:

$$\lambda^v = \pm\sqrt{\frac{\nu_v}{T_v}}, \quad \pm\sqrt{\frac{\nu_h}{T_h}}, \quad 0$$

Viscous and heating waves



First-Order NS = *Hyperbolic* Inviscid term + *Hyperbolic* Viscous term

Methods for hyperbolic systems are all we need.

Hyperbolic Navier-Stokes System

$$\mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S}$$

Intrinsic Features:

1. Methods for hyperbolic systems apply to the whole system.
2. *O(1/h)* faster than 'traditional' Navier-Stokes codes.
3. Accurate viscous/heat fluxes, of course.

If you have a good method for the Euler,
you have a very good method for the Navier-Stokes.

High-order, FVS, Riemann solver, positive or non-oscillatory, implicit, multigrid….

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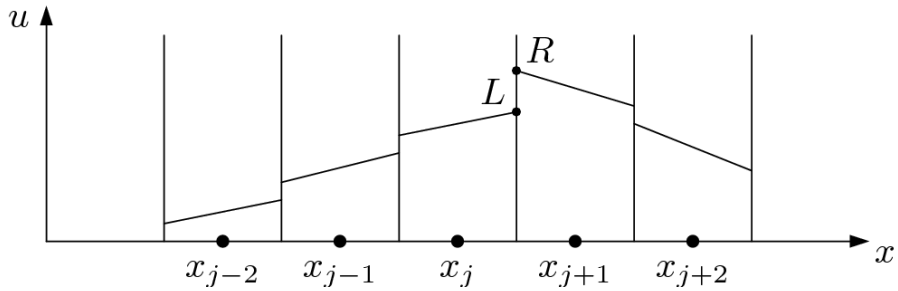
Hyperbolic Navier-Stokes Scheme



Finite-Volume Method:

$$\mathbf{P}_j^{-1}\frac{d\mathbf{U}_j}{dt} = -\frac{1}{\Delta x}\left[\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right] + \mathbf{S}_j$$

Upwind Navier-Stokes Flux:

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left[\mathbf{F}_R + \mathbf{F}_L\right] - \frac{1}{2}\mathbf{P}^{-1}\left|\mathbf{PA}\right|\left(\mathbf{U}_R - \mathbf{U}_L\right)$$

where

$$\left|\mathbf{PA}\right| \approx \left|\mathbf{PA}^i\right| + \left|\mathbf{PA}^v\right|$$

That's it. You can now start writing a Navier-Stokes code.

O(h) Time Step

Hyperbolic Navier-Stokes Scheme (Nishikawa, AIAA2011-3043) :

$$\Delta t = \mathrm{CFL}\frac{h}{\max(|u| + a + a_h)_j} = \boxed{O(h)}$$

Traditional Navier-Stokes Scheme (Nishikawa, AIAA2011-3044) :

$$\Delta t = \mathrm{CFL}\frac{h}{\max(|u| + a + O(1/h))_j} = \boxed{O(h^2)}$$

O(1/h) speed-up to reach the steady state (1/2/3D).

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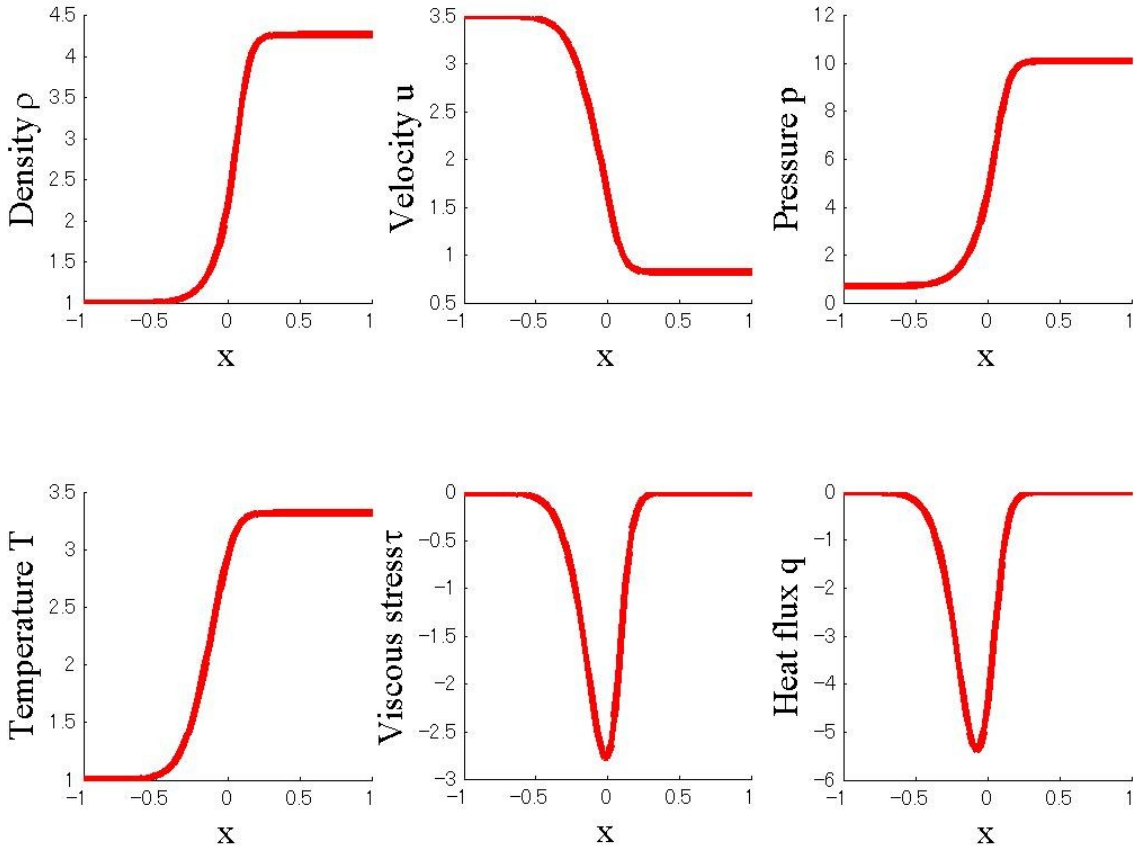
Viscous Shock Structure Problem

M_∞=3.5, Pr=$\frac{3}{4}$, γ=1.4, Re_∞=25, T_∞=400$[K]$.



Program can be downloaded at http://www.cfdbooks.com

2D Numerical Results

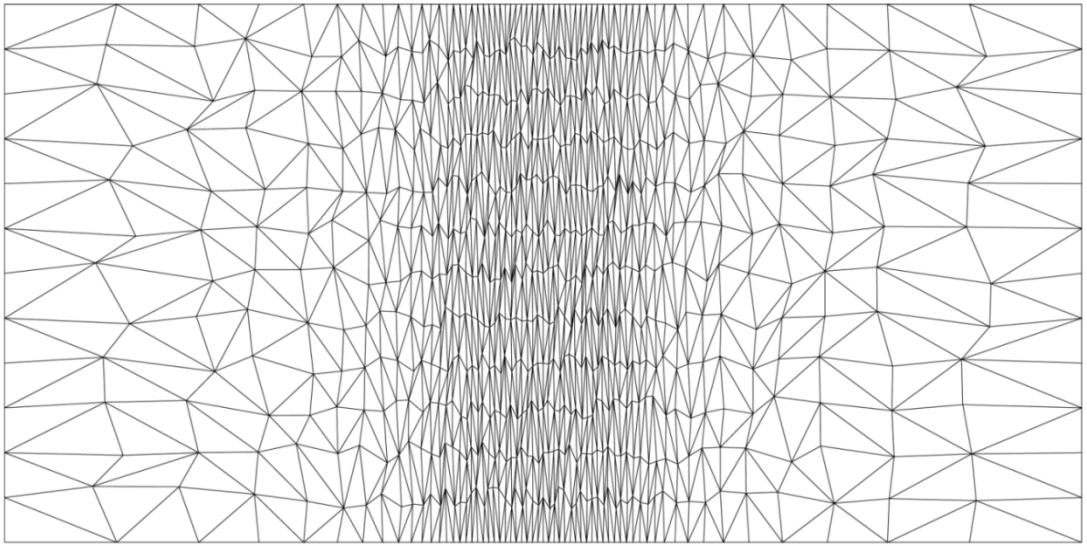
Viscous shock structure problem in 2D domain

Node-centered edge-based FV method

1. *Hyperbolic* scheme (Upwind Navier-Stokes flux) 9 equations
2. *Traditional* scheme (Roe flux, Ave-LSQ flux) 4 equations

Steady state is reached at residual reduction by 6 orders.

Irregular triangular grids: 533, 867, 1281, 1751, 2349, nodes.



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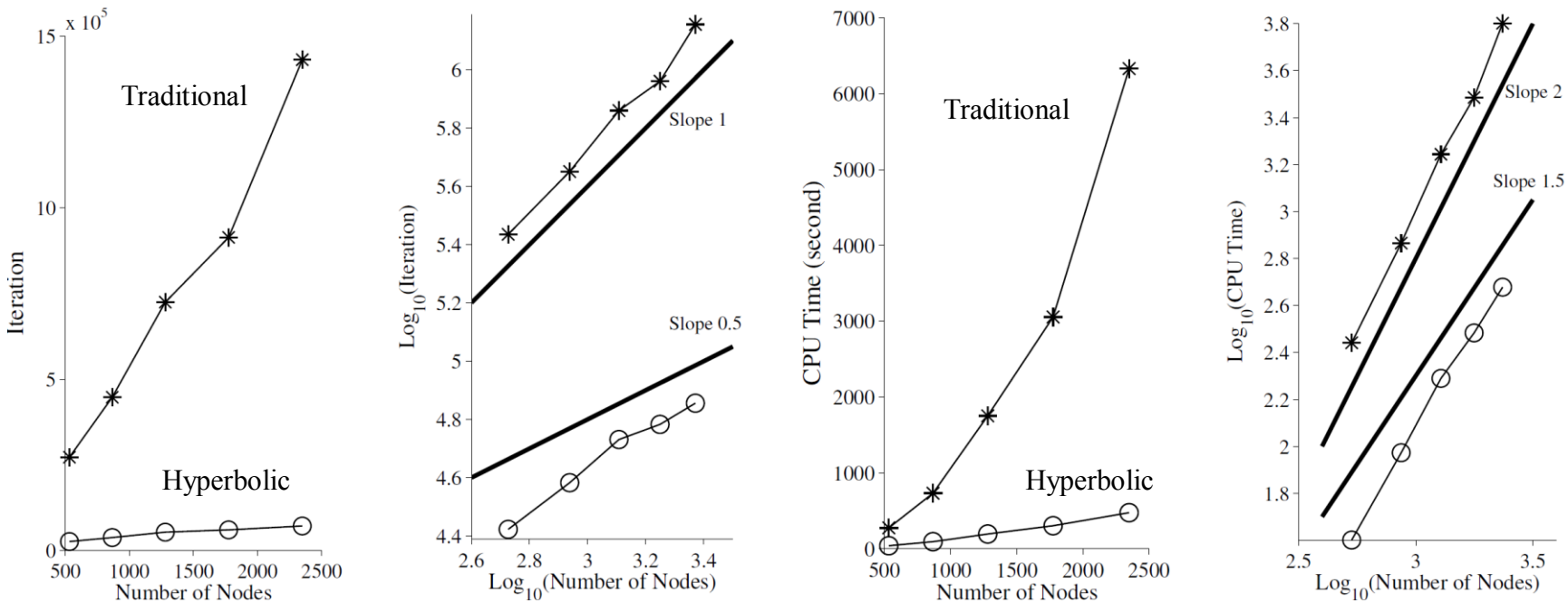
Convergence Results



(a) Iteration vs. N (b) Log-log plot of (a). (c) CPU time vs. N. (d) Log-log plot of (c).

Hyperbolic scheme is O(1/h) faster in CPU time.

Cost of solving extra equations is overwhelmed by O(1/h) speed-up.

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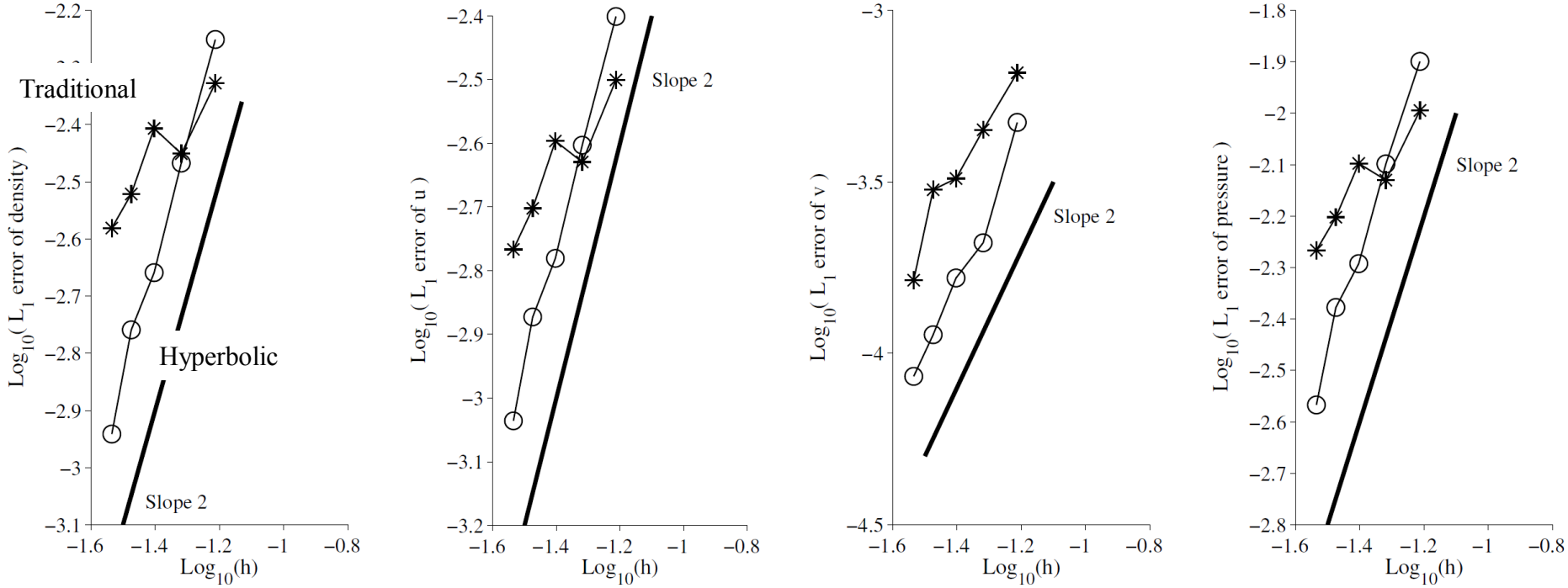
Error Convergence in Main Variables



(a) L_1 error of ρ. (b) L_1 error of u. (c) L_1 error of v. (d) L_1 error of p.

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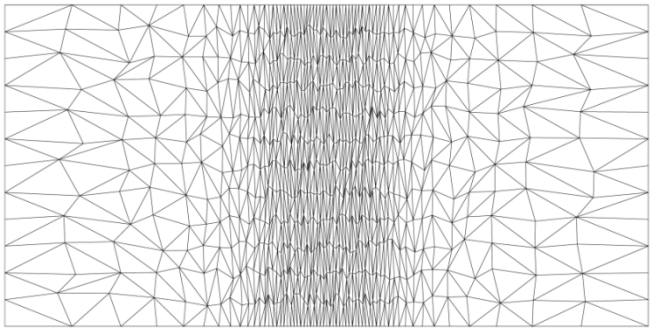
Error Convergence in Viscous/Heat Fluxes

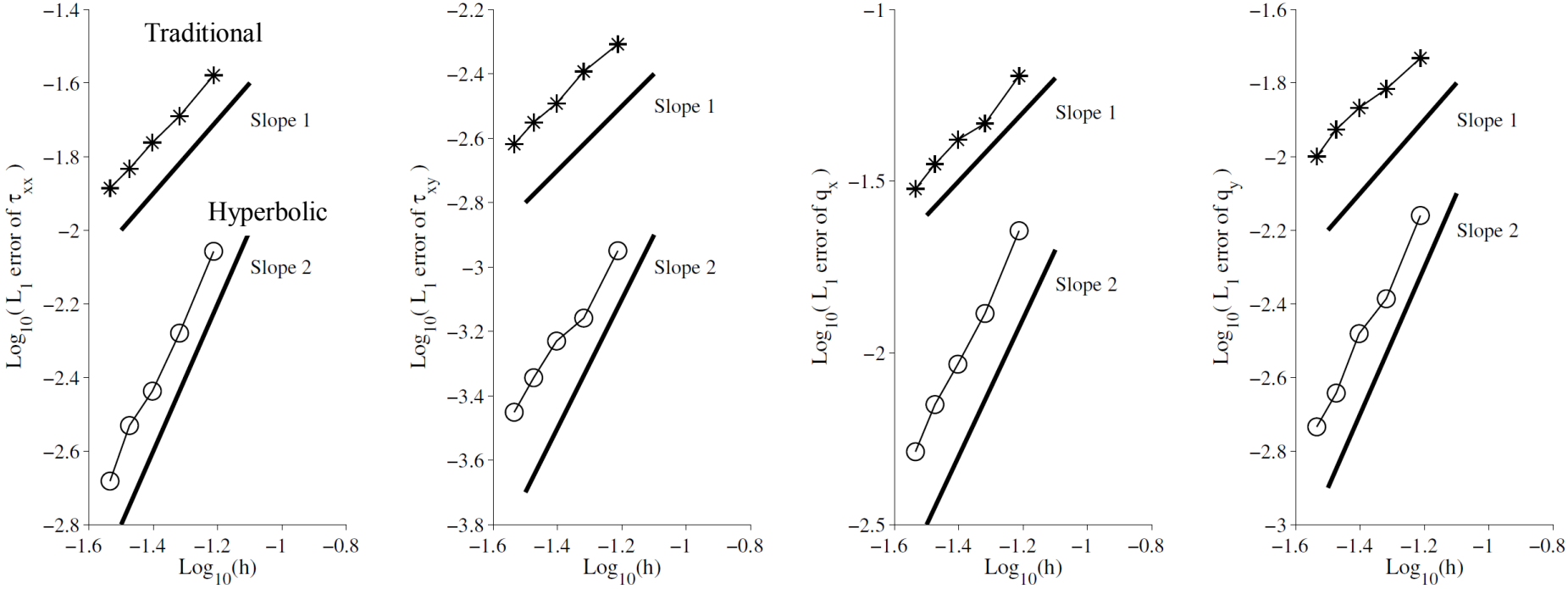
1. Hyperbolic Scheme:

   Simultaneously computed.

2. Traditional Scheme:

   LSQ reconstruction





(a) L_1 error of τ_{xx}. (b) L_1 error of τ_{xy}. (c) L_1 error of q_x. (d) L_1 error of q_y.

First-Order Hyperbolic System Method

Construct a hyperbolic system for target PDEs, and integrate it towards the steady state.

Demonstrated for Diffusion, Advection-Diffusion, and NS Equations:

Upwind scheme for viscous term
O(1/h) speed-up by explicit scheme
Accurate viscous/heat fluxes on irregular grids

Accurate viscous force and heating rate prediction on unstructured grids come with orders-of-magnitude improvement in efficiency.

The door is now open for applications to various nonlinear systems.

Download papers, slides, and codes at http://www.hiroakinishikawa.com

Lots of Lots of Future Work

Non-Oscillatory Schemes – *High-Reynolds, shock waves.*

Implicit Schemes – *Simpler construction, fast iterative linear solvers*

Time-Accurate Schemes – *Dual-time stepping with fast inner iterations.*

Other Discretization Methods – *FV, FE, CESE, RD, DG, SV, SD,.....*

Other Hyperbolic NS Systems – *Include vorticity or solution gradients.*

Turbulence Models – *All gradients in source terms turned into scalars.*

Other Governing Eqns – *Incomp. NS, Turbulence, Resistive MHD...*

Higher-Order PDEs – *Construct a hyperbolic system and integrate it.*

And so on and on and on...

Hyperbolic Viscous System is a Coin.

This presentation showed only a portion of a side of the coin (red square).

On the other side of the coin lie a variety of traditional viscous schemes – 2nd-derivatives:
The hyperbolic viscous system generates also robust and accurate traditional viscous schemes.

That's another story about challenging the common wisdom on viscous discretization.

Why is it a dime?
Because I put my two cents in 5 papers: JCP2007/2010, Computers&Fluids2011, AIAA2011-3043/3044

Hyperbolic Schemes:
JCP2007/2010, AIAA2011-3043

Traditional Schemes:
Compters&Fluids2011, AIAA2011-3044