Hyperbolic Method for Dispersive PDEs: Same High-Order of Accuracy for Solution, Gradient, and Hessian

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Background

Consider the following PDE

\[
\partial_t u + \partial_x F(u) = D
\]

Hyperbolic model
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*Hyperbolic model*

1) Classical viscous regularization \( \rightarrow D = \nu \partial_{xx} u \)
Background

Consider the following PDE

\[ \partial_t u + \partial_x F(u) = \mathcal{D} \]

**Hyperbolic model**

1) Classical viscous regularization → \( \mathcal{D} = \nu \partial_{xx} u \)
2) Dispersive with time derivative → \( \mathcal{D} = \epsilon \partial_{xxt} u \)

The original hyperbolic advection-diffusion formulation \([\text{Hiro, JCP 227}(2007), \text{315–352}]\) will not work for this PDE! Why? Because the resulting system will not be hyperbolic \([\text{shown first by Toro and Montecinos, SIAM JSC 36(2014), A2423–A2457}]\)
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2) Dispersive with time derivative \( \rightarrow D = \epsilon \partial_{xxt} u \)

The PDE can then be decomposed to \( \Rightarrow \)

\[ \begin{align*}
\partial_t w &= -\partial_x f \\
-\partial_{xx} u + u &= w
\end{align*} \]
Consider the following PDE
\[
\frac{\partial_t u}{\partial x} + \frac{\partial_x F(u)}{\partial x} = \mathcal{D}
\]

Hyperbolic model

1) Classical viscous regularization \( \rightarrow \mathcal{D} = \nu \partial_{xx} u \)

2) Dispersive with time derivative \( \rightarrow \mathcal{D} = \epsilon \partial_{xxt} u \)

The PDE can then be decomposed to \( \Rightarrow \)
\[
\begin{align*}
\partial_t w &= -\partial_x f \\
-\partial_{xx} u + u &= w
\end{align*}
\]

- We have already proposed very accurate numerical schemes for cases (1) and (2); e.g., [Hiro, JCP 227(2007), 315–352], [Alireza & Hiro, NASA/TM-2014-218174], [Alireza & Hiro, CF 102 (2014), 131–147], [Hiro, JCP 281(2015), 518–555], [Alireza and Hiro JCP 300(2015), 455–491]
Background

Consider the following PDE

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1) Classical viscous regularization \( \rightarrow \mathcal{D} = \nu \partial_{xx} u \)
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3) Fully Dispersive \( \rightarrow \mathcal{D} = \epsilon \partial_{xxx} u \)

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- The original hyperbolic advection-diffusion formulation [Hiro, JCP 227(2007), 315–352] will not work for this PDE! *Why?*
Consider the following PDE

$$\partial_t u + \partial_x F(u) = D$$

*Hyperbolic model*

1) Classical viscous regularization $\rightarrow D = \nu \partial_{xx} u$

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3) Fully Dispersive $\rightarrow D = \epsilon \partial_{xxx} u$

- The original hyperbolic advection-diffusion formulation [Hiro, JCP 227(2007), 315–352] will not work for this PDE! Why?
- Because the resulting system will not be hyperbolic [shown first by Toro and Montecinos, SIAM JSC 36(2014), A2423–A2457]
Consider a Fully Dispersive PDE (the 3rd case)

\[
\frac{\partial_t u}{\text{physical time}} = \epsilon \frac{\partial_{xxx} u}{\text{dispersion}}
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Consider a Fully Dispersive PDE (the 3rd case)

\[
\partial_t u = \varepsilon \partial_{xxx} u
\]

Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)
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Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

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0 = -\alpha \frac{u}{\Delta t} + s
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Consider a Fully Dispersive PDE (the 3rd case)

\[ \partial_t u = \epsilon \partial_{xxx} u \]

Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

\[ 0 = \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s \]
Consider a Fully Dispersive PDE (the 3rd case)

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\frac{\partial_t u}{\text{physical time}} = \epsilon \frac{\partial_{xxx} u}{\text{dispersion}}
\]

Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

\[
0 = \epsilon \frac{\partial_x q}{\partial \tau} - \alpha \frac{u}{\Delta t} + s
\]

\[
0 = \frac{1}{T_e} (\partial_x u - p) \quad \Rightarrow p = u_x
\]

\[
0 = \frac{1}{T_e} (\partial_x p - q) \quad \Rightarrow q = p_x = u_{xx}
\]
Consider a Fully Dispersive PDE (the 3rd case)

\[
\begin{align*}
\partial_t u &= \epsilon \partial_{xxx} u \\

\end{align*}
\]

Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

\[
\begin{align*}
\partial_\tau u &= \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s \\
\partial_\tau p &= \frac{1}{T_\epsilon} (\partial_x u - p) \Rightarrow p = u_x \\
\partial_\tau q &= \frac{1}{T_\epsilon} (\partial_x p - q) \Rightarrow q = p_x = u_{xx} \\

\end{align*}
\]

→ in a vector form:

\[
\begin{align*}
\frac{\partial U}{\partial \tau} + \frac{\partial F}{\partial x} &= Q \\
\text{pseudo time} &+ \text{physical time} \text{ all sources}
\end{align*}
\]
Hyperbolic or Not Hyperbolic

- From the previous slide, \( \partial_t u = \epsilon \partial_{xxx} u \)
  \[ \rightarrow \text{in a vector form:} \]
  \[
  \frac{\partial U}{\partial \tau} + \frac{\partial F}{\partial x} = Q
  \]
  \[ ? \]

\[ \begin{bmatrix}
  0 & 0 \\
  -\epsilon & -1 \\
  \epsilon & 0 \\
  0 & -1
  \end{bmatrix},
  U = \begin{bmatrix} u \\ p \to u \\ x \\ q \to u \\ xx \end{bmatrix} \]

Eigenvalues:
\[ \lambda_1 = -\frac{\epsilon}{3}, \lambda_2, 3 = \frac{2}{3} \lambda_1 (1 \pm \sqrt{3}i) \]
⇒ Not Hyperbolic!
Hyperbolic or Not Hyperbolic

From the previous slide, \( \frac{\partial_t u}{\partial t} = \epsilon \frac{\partial_{xxx} u}{\partial x} \)

→ in a vector form:

\[ \begin{align*}
\frac{\partial \mathbf{U}}{\partial \tau} + \frac{\partial \mathbf{F}}{\partial x} &= \mathbf{Q} \\
\frac{\partial \mathbf{F}}{\partial x} &= \mathbf{A} \frac{\partial \mathbf{U}}{\partial x}, \quad \mathbf{A} = \begin{bmatrix}
0 & 0 & -\epsilon \\
-\frac{1}{T_\epsilon} & 0 & 0 \\
0 & -\frac{1}{T_\epsilon} & 0
\end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix}
u \\
p \rightarrow u_x \\
q \rightarrow u_{xx}
\end{bmatrix}
\end{align*} \]

Eigenvalues:
\[ \lambda_1 = -\frac{1}{3} \epsilon, \lambda_2, \lambda_3 = \frac{1}{2} \lambda_1 (1 \pm \sqrt{3}i) \]
⇒ Not Hyperbolic!
Hyperbolic or Not Hyperbolic

From the previous slide, \( \partial_t u = \epsilon \partial_{xxx} u \)

→ in a vector form:

\[
\frac{\partial U}{\partial \tau} + \frac{\partial F}{\partial x} = Q
\]

\[
\frac{\partial F}{\partial x} = A \frac{\partial U}{\partial x}, \quad A = \begin{bmatrix}
0 & 0 & -\epsilon \\
-\frac{1}{T_\epsilon} & 0 & 0 \\
0 & -\frac{1}{T_\epsilon} & 0
\end{bmatrix}, \quad U = \begin{bmatrix}
u \\
p \rightarrow u_x \\
q \rightarrow u_{xx}
\end{bmatrix}
\]

Eigenvalues:

\[
\lambda_1 = -\epsilon^{1/3} T_\epsilon^{-2/3}, \quad \lambda_{2,3} = \frac{1}{2} \lambda_1 (1 \pm \sqrt{3} I)
\]
**Hyperbolic or Not Hyperbolic**

- From the previous slide, \( \partial_t u = \epsilon \partial_{xxx} u \)
  \[ \rightarrow \text{in a vector form:} \]

\[
\begin{align*}
\frac{\partial U}{\partial \tau} + \frac{\partial F}{\partial x} &= \mathbf{Q} \\
\frac{\partial F}{\partial x} &= \mathbf{A} \frac{\partial U}{\partial x}, \quad \mathbf{A} = \begin{bmatrix}
0 & 0 & -\epsilon \\
- \frac{1}{T_\epsilon} & 0 & 0 \\
0 & - \frac{1}{T_\epsilon} & 0
\end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix}
u \\
p \rightarrow u_x \\
q \rightarrow u_{xx}
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\]

- Eigenvalues:

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\lambda_1 = -\epsilon^{1/3} T_\epsilon^{-2/3}, \quad \lambda_{2,3} = \frac{1}{2} \lambda_1 (1 \pm \sqrt{3} I) \quad \Rightarrow \quad \text{Not Hyperbolic!}
\]
Consider (again) a Fully Dispersive PDE (the 3rd case)

\[
\partial_t u = \epsilon \partial_{xxx} u
\]
Consider (again) a Fully Dispersive PDE (the 3rd case)

\[
\frac{\partial_t u}{
\text{physical time}
} = \epsilon \frac{\partial_{xxx} u}{
\text{dispersion}
}
\]

- Introducing a New Reformulation Strategy
Consider (again) a Fully Dispersive PDE (the 3rd case)

\[
\partial_t u = \epsilon \partial_{xxx} u
\]

Introducing a New Reformulation Strategy

\[
0 = -\alpha \frac{u}{\Delta t} + s
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Consider (again) a Fully Dispersive PDE (the 3rd case)

\[ \partial_t u = \epsilon \partial_{xxx} u \]

Introducing a New Reformulation Strategy

\[ 0 = \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s \]
Consider (again) a Fully Dispersive PDE (the 3rd case)

\[
\partial_t u = \epsilon \partial_{xxx} u
\]

Introducing a New Reformulation Strategy

\[
\begin{align*}
0 &= \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s \\
0 &= \frac{1}{T\epsilon} (\partial_x u - p) \\
0 &= \frac{1}{T\epsilon} (\partial_x p - q)
\end{align*}
\]
Consider (again) a Fully Dispersive PDE (the 3rd case)

\[
\partial_t u = \epsilon \partial_{xxx} u
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Introducing a New Reformulation Strategy

\[
\begin{align*}
\partial_\tau u &= \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s \\
\partial_\tau p &= \frac{1}{T_\epsilon} (\partial_x u - p) \\
\partial_\tau q &= \frac{1}{T_\epsilon} (\partial_x p - q)
\end{align*}
\]
Consider (again) a Fully Dispersive PDE (the 3rd case)

\[
\frac{\partial_t u}{\text{physical time}} = \epsilon \frac{\partial_{xxx} u}{\text{dispersion}}
\]

Introducing a New Reformulation Strategy

\[
\frac{\partial_{\tau} u}{\text{pseudo time}} = \epsilon \frac{\partial_x q}{\text{}} - \alpha \frac{u}{\Delta t} + s
\]

\[
\frac{\partial_{\tau} p}{\text{}^}\frac{1}{T\epsilon} \left( \frac{\partial_x u}{\text{}} - p - \gamma u \right) \Rightarrow p = u_x - \gamma u
\]

\[
\frac{\partial_{\tau} q}{\text{}} = \frac{1}{T\epsilon} \left( \frac{\partial_x p}{\text{}} - q \right)
\]
Consider (again) a Fully Dispersive PDE (the 3rd case)

\[ \partial_t u = \epsilon \partial_{xxx} u \]

Introducing a New Reformulation Strategy

\[ \partial_\tau u = \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s \]
\[ \partial_\tau p = \frac{1}{T_\epsilon} (\partial_x u - p - \gamma u) \implies p = u_x - \gamma u \]
\[ \partial_\tau q = \frac{1}{T_\epsilon} (\partial_x p - q + \gamma u_x) \implies q = p_x + \gamma u_x = u_{xx} \]
Consider (again) a Fully Dispersive PDE (the 3rd case)

Introducing a New Reformulation Strategy

\[ \partial_\tau u = \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s \]
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→ in a vector form:

\[ \frac{\partial U}{\partial \tau} + \frac{\partial F}{\partial x} = Q \]

pseudo time + physical time + all sources
New Formulation: Is it Hyperbolic? How?

- \( \gamma \neq 0 \) is critical

We found...

\[ \gamma = \beta + \frac{1}{\epsilon \beta^2 T_\epsilon}, \quad \beta = \frac{\kappa}{L_\epsilon} \]

- The New system has the following Eigenvalues:

\[ \lambda_1 = \frac{1}{\beta T_\epsilon}, \quad \lambda_{2,3} = \frac{\lambda_1}{2} \left( -1 \pm \sqrt{1 + 4\epsilon \beta^3 T_\epsilon} \right) \]

- and the following independent Eigenvectors:

\[
\mathbf{R} = \begin{bmatrix}
-\epsilon & -\lambda_2 T_\epsilon & -\lambda_3 T_\epsilon \\
\epsilon \beta & 1 & 1 \\
\lambda_1 & \beta - \lambda_2/(\epsilon \beta) & \beta - \lambda_3/(\epsilon \beta)
\end{bmatrix} \Rightarrow \text{Is Hyperbolic!} \]
Dispersion Relaxation Time

\[ T_\epsilon = \frac{Dispersion \ Length}{Wave \ Length} = \frac{L_\epsilon}{\lambda_2} \]

- Note \( \lambda = \lambda(\beta), \quad \beta = \beta(\kappa) \)

Choice of Arbitrary Constant \( \kappa \)

- any value except \( \kappa = 1 \) is a valid value
- \( \kappa = 1 \rightarrow \) singular eigenvector
- we recommend

\[ \kappa = \kappa_g \equiv \frac{1 + \sqrt{5}}{2} \]

which makes everything simpler:

\[ T_\epsilon = \frac{L_\epsilon^3}{\epsilon} \]
New Hyperbolic System for Dispersion

\[\lambda_1 = \frac{L_\epsilon}{\kappa_g T_\epsilon}, \quad \lambda_2 = \kappa_g \lambda_1, \quad \lambda_3 = -\kappa_g^2 \lambda_1,\]

The \( \gamma \) parameter can now be fully defined as

\[\gamma = \beta + \frac{1}{\epsilon \beta^2 T_\epsilon} = \frac{2}{L_\epsilon}\]
Dispersion Length Scale

- **Fourier Analysis:** 
  \[ U = U_0 e^{i \theta h} \]  
  \[ \frac{\partial U}{\partial \tau} + \frac{\partial F}{\partial x} = Q \]

- **Eigenvalues of Fourier-Transformed Operator:**
  \[ \lambda_1 = -\frac{I \epsilon \theta^3}{h^3} + \frac{2I \epsilon L^2_\epsilon}{h^5} \theta^5 + \frac{2\epsilon L^3_\epsilon}{h^6} \theta^6 + O(\theta^7), \]
  \[ \lambda_{2/3} = -\frac{\epsilon}{L^3_\epsilon} \pm \frac{I \sqrt{2} \epsilon}{L^2_\epsilon h} \theta + O(\theta^2). \]

- **Damping Mode**

- **Propagation Mode**

- **For the slowest frequency error mode (\( \theta = \pi h \)):**
  \[ \frac{\epsilon}{h^3} \theta^3 = \frac{\sqrt{2} \epsilon}{L^2_\epsilon h} \theta \Rightarrow L_\epsilon = \pm \frac{2^{\frac{1}{4}}}{\pi} \]
Hyperbolic Advection-Diffusion-Dispersion

Consider a general Advective-Diffusive-Dispersive PDE

\[
\begin{align*}
\partial_t u + \partial_x (f) &= \partial_x (\nu \partial_x u) + \epsilon \partial_{xxx} u \\
\partial_t p &= \partial_x u - \gamma u \\
\partial_t q &= \frac{1}{T} \epsilon \left( \partial_x p - q + \gamma u x \right)
\end{align*}
\]
Consider a general Advective-Diffusive-Dispersive PDE

\[
\begin{align*}
\frac{\partial}{\partial \tau} u &= \frac{1}{T_\epsilon} \left( \frac{\partial}{\partial x} p - q + \gamma u_x \right) \\
\frac{\partial}{\partial \tau} p &= \left( \frac{1}{T_\epsilon} \right) \left( \frac{\partial}{\partial x} u - p - \gamma u \right) \\
\frac{\partial}{\partial \tau} q &= \frac{1}{T_\epsilon} \left( \frac{\partial}{\partial x} p - q + \gamma u_x \right)
\end{align*}
\]

\[\Rightarrow p = \frac{\partial}{\partial x} u - \gamma u, \quad q = p_x + \gamma u_x = u_{xx}\]
### Hyperbolic Advection-Diffusion-Dispersion

Consider a general Advective-Diffusive-Dispersive PDE

$$\begin{align*}
\partial_t u &+ \partial_x(f) = \partial_x(\nu \partial_x u) + \epsilon \partial_{xxx} u \\
\partial_\tau u &= -\partial_x(f) \\
\partial_\tau p &= \left(\frac{1}{T_\epsilon}\right) (\partial_x u - p - \gamma u) \Rightarrow p = \partial_x u - \gamma u \\
\partial_\tau q &= \frac{1}{T_\epsilon} (\partial_x p - q + \gamma u_x) \Rightarrow q = p_x + \gamma u_x = u_{xx}
\end{align*}$$
Consider a general Advective-Diffusive-Dispersive PDE

\[
\partial_t u = \partial_x(f) + \partial_x(\nu \partial_x u) + \epsilon \partial_{xxx} u
\]

\[
\partial_\tau u = -\partial_x(f) + \partial_x(\nu p + \gamma \nu u) + \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s
\]

\[
\partial_\tau p = \left( \frac{1}{T_\epsilon} \right) (\partial_x u - p - \gamma u) \quad \Rightarrow \quad p = \partial_x u - \gamma u
\]

\[
\partial_\tau q = \frac{1}{T_\epsilon} (\partial_x p - q + \gamma u_x) \quad \Rightarrow \quad q = p_x + \gamma u_x = u_{xx}
\]
Consider a general Advective-Diffusive-Dispersive PDE

\[
\begin{align*}
\partial_t u &= -\partial_x (f) + \partial_x (\nu p + \gamma \nu u) + \epsilon \partial_x q - \alpha \frac{u}{\Delta t} + s \\
\partial_\tau p &= \left(\frac{1}{T_\epsilon} + \frac{1}{T_\nu}\right)(\partial_x u - p - \gamma u) \\
\partial_\tau q &= \frac{1}{T_\epsilon}(\partial_x p - q + \gamma u_x)
\end{align*}
\Rightarrow p = \partial_x u - \gamma u
\Rightarrow q = p_x + \gamma u_x = u_{xx}
\]
Discretization: Residual-Distribution

Cell Residual

\[ \Phi^E = \int_j^{j+1} (-A \partial_x U + Q) \, dx \]

Residual Distribution

\[ \Phi^L = B^- \Phi^E, \quad \Phi^R = B^+ \Phi^E \]
Discretization: Residual-Distribution

- SUPG Distribution

\[
B^\pm = \frac{1}{2} I \pm D^{\text{adv}} \pm D^{\text{diff}} \pm D^{\text{disp}}
\]

\[
B^\pm = \frac{1}{2} \begin{bmatrix} 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \\ \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \\ \end{bmatrix}
\]

\[
= \frac{1}{2} I \pm \frac{1}{2} A \left( R |\Lambda|^{-1} L \right) = \frac{1}{2} I \pm \frac{1}{2} A \sum_{l=1}^{3} \frac{1}{|\lambda_l|} \Pi_l,
\]

\[\text{Galerkin Stabilization}\]
Discretization: Residual-Distribution

- **SUPG Stabilization**

\[
\begin{align*}
\mathbf{B}^{\pm} &= \frac{1}{2} \mathbf{I} \\
\text{Residual Distribution} &= \pm \mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}} \\
\text{Galerkin} &= \pm \frac{1}{2} \mathbf{A} \sum_{l=1}^{3} \frac{1}{|\lambda_l|} \Pi_l \\
\text{Stabilization} &= \end{align*}
\]
Discretization: Residual-Distribution

SUPG Stabilization

\[ B^\pm = \frac{1}{2} I \pm D_{\text{adv}} \pm D_{\text{diff}} \pm D_{\text{disp}} \]

\[ = \frac{1}{2} I \pm \sum_{l=1}^{3} \frac{1}{|\lambda_l|} \Pi_l \]

\[ A_{\text{adv}} + A_{\text{diff}} + A_{\text{disp}} \]
Discretization: Residual-Distribution

- **SUPG Stabilization**

\[
\mathbf{B}^\pm = \frac{1}{2} \mathbf{I} \pm \mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}}
\]

- **Residual Distribution**

\[
\mathbf{B}^\pm = \frac{1}{2} \mathbf{I} \pm \mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}}
\]

- **Galerkin**

\[
\mathbf{B}^\pm = \frac{1}{2} \mathbf{I} \pm \mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}}
\]

- **Stabilization**

\[
\mathbf{B}^\pm = \frac{1}{2} \mathbf{I} \pm \mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}}
\]

\[
\Rightarrow \text{Individual Stabilization terms are:}
\]

\[
\mathbf{D}^{\text{adv}} = \frac{1}{2} \mathbf{A}^{\text{adv}} / (|a - \gamma u| + \tilde{\epsilon}), \quad \tilde{\epsilon} \ll 1
\]

\[
\mathbf{D}^{\text{diff}} = \frac{1}{2} \mathbf{A}^{\text{diff}} \sum_{l=1}^{2} \frac{1}{|\lambda_l^{\text{diff}}|} \Pi_l^{\text{diff}}, \quad \mathbf{D}^{\text{disp}} = \frac{1}{2} \mathbf{A}^{\text{disp}} \sum_{l=1}^{3} \frac{1}{|\lambda_l^{\text{disp}}|} \Pi_l^{\text{disp}}
\]
Two-Soliton Korteweg-de Vries (KdV)

Consider a general time-dependent dispersive PDE

\[
\partial_t u + \partial_x f = \partial_x (\nu \partial_x u) + \epsilon \partial_{xxx} u + \tilde{s}(x, u), \quad f = 3u^2, \quad \nu = 0.5, \quad \epsilon = 1
\]

Method of Manufactured Solution

\[
u^e(x, t) = \frac{(\eta_2 - \eta_1) \left( \eta_1 \sech^2 [\chi(\eta_1)] + \eta_2 \csch^2 [\chi(\eta_2)] \right)}{(\sqrt{\eta_1} \tanh [\chi(\eta_1)] - \sqrt{\eta_2} \coth [\chi(\eta_2)])^2},
\]

\[
\chi(\eta) = \sqrt{\frac{\eta}{2}} (x - 2 \eta t - \tilde{a})
\]

\[
\eta_1 = 0.5, \quad \eta_2 = 1.0, \quad \tilde{a} = 1.0
\]

Boundary Condition & Grid

Dirichlet Boundary with Randomly Distributed Grid Points
Two-Soliton Korteweg-de Vries (KdV):
Fourth-Order RD Scheme [Alireza & Hiro, CF 102(2014), 131–147]
on $N = 60 \in x[0, 30]$ with BDF2, $\Delta t = 0.01$

$t = 1$

$t = 4$

$t = 8$

$t = 12$
Two-Soliton Korteweg-de Vries (KdV):

on $N = 60 \in x[0, 30]$ with BDF2, $\Delta t = 0.001$

Gradient, $u_x(x, t = 10)$

Hessian, $u_{xx}(x, t = 10)$

4th-order scheme accurately detects the peaks and valleys.
Accuracy Verification on Randomly Distributed Grid Points

Temporal accuracy (BDF2), \( N = 640 \)

Spatial accuracy (fourth-order), \( \Delta t = 0.001 \)

- Equal order of accuracy (4th-order) for \( u, u_x \) & \( u_{xx} \)
Korteweg-de Vries (KdV)

Consider a time-dependent KdV equation

$$\partial_t u + \partial_x f = \epsilon \partial_{xxx} u, \quad f = \frac{u^2}{2}, \quad \epsilon = -4.84 \times 10^{-4}$$

Initial Condition

$$u(x, 0) = 3 \eta_1 \text{sech}^2 \left( \frac{1}{2} \sqrt{-\eta_1/\epsilon} \left[ (x - x_1) - \eta_1 t \right] \right)$$

$$+ 3 \eta_2 \text{sech}^2 \left( \frac{1}{2} \sqrt{-\eta_2/\epsilon} \left[ (x - x_2) - \eta_2 t \right] \right)$$

$$\eta_1 = 0.3, \quad \eta_2 = 0.1, \quad x_1 = 0.4, \quad x_2 = 0.8$$

Boundary Condition & Grid

Periodic Boundary with Randomly Distributed Grid Points
Korteweg-de Vries (KdV): Solution $u$

$t = 0$

$t = 1$

$t = 2$

$t = 3$
Korteweg-de Vries (KdV): Solution Gradient $u_x$

$t = 0$

$t = 1$

$t = 2$

$t = 3$
Korteweg-de Vries (KdV): Hessian $u_{xx}$

$t = 0$

$t = 1$

$t = 2$

$t = 3$
Dispersive Shock (Zero Dispersion Limit)

Consider a time-dependent Dispersive Nonlinear Burgers equation

\[ \partial_t u + u \partial_x u = \epsilon \partial_{xxx} u, \quad \epsilon \to 0^+ \]

Behavior of this equation as \( \epsilon \to 0^+ \)

- Continuous wavelets in the vicinity of the discontinuity (aka Dispersive Shock Waves)
- The smaller the \( \epsilon > 0 \), the higher the frequency of the wavelets
- Known as an extremely difficult case
Zero Dispersion Limit: $\epsilon = 10^{-4}$

Solution, $u$

Gradient, $u_x$

Hessian, $u_{xx}$
Zero Dispersion Limit: $\epsilon = 10^{-4}$

Solution, $u$

Gradient, $u_x$

Hessian, $u_{xx}$

Zero Dispersion Limit: $\epsilon = 10^{-5}$

Solution, $u$

Gradient, $u_x$

Hessian, $u_{xx}$
Summary

- Introduced a NEW first-order system for general Dispersive PDEs
- Showed that the proposed system is hyperbolic (in pseudo-time)
- Illustrated that high-order scheme applied to the proposed system produces equal order of accuracy for solution, gradient and Hessian (second-derivatives)
- Demonstrated the capability of the developed high-order schemes in accurately capturing the dispersive shocks
High-Order DG Schemes with the First-Order Hyperbolic System Approach  

- Same DoF as the conventional DG
- \((k + 2)\)-order accurate solution & \((k + 1)\)-order accurate gradients for Advection problems
- \((k + 1)\)-order accurate solution & gradients for Diffusion problems

Next Talk: To be presented in April/May 2016!
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