Polynomial Chaos Decomposition with Differentiation and Applications

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Outline

- Introduction
- Theory and Methodology
- Applications
  - A. Analytical Problems
    - i. One-dimensional problem
    - ii. Two-dimensional problem
  - B. Composite Laminate Problems
    - i. Macroscale Modeling of a CFRP Composite Laminate
    - ii. Multiscale Modeling of a CFRP Composite Laminate
- Conclusion
- Future work
Why Uncertainty Quantification (UQ)?

PC expansion: spectral method for Uncertainty Quantification \( (L_2 \text{ Convergence}) \)

Methods for PC expansion:
- i) Intrusive, and ii) Non-intrusive

PCDD is a new non-intrusive technique that involves differentiation of the multivariate polynomials and sensitivity calculation

Number of samples equal to number of PC expansion coefficients
Polynomial Chaos Expansion

- PC expansion (introduced by Norbert Weiner, 1938) of uncertain response, \( y = f(x) \):
  \[
y(\tilde{\xi}) = \sum_{k=0}^{P} a_k \psi_k(\tilde{\xi}); \quad \text{Using NQ: } a_k = \frac{\langle y, \psi_k(\tilde{\xi}) \rangle}{\langle \psi_k(\tilde{\xi}), \psi_k(\tilde{\xi}) \rangle}
\]

- To obtain \( y \), find PC expansion coefficients, \( a_k \)
- Express input and response random variables in PC expansion form:
  \[
y(\tilde{x}) = y(\tilde{\xi}) = \sum_{k=0}^{P} a_k \psi_k(\tilde{\xi}); \quad x_i = \sum_{j=0}^{t} \lambda_{j1} \psi_i(\xi_i)
\]
- Substitute in \( y = f(\tilde{x}) \):
  \[
  \sum_{k=0}^{P} a_k \psi_k(\tilde{\xi}) = f(\sum_{j=0}^{t} \lambda_{j1} \psi_j(\xi_1), \ldots, \sum_{j=0}^{t} \lambda_{jd} \psi_j(\xi_d))
  \]
Non Intrusive PC Expansion

- Responses are generated for different realizations of the uncertain inputs
- Software such as ABAQUS, NASTRAN/PATRAN, ANSYS can be used to get the response samples
Polynomial Chaos Expansion

Integration ➔ Sampling

Differentiation ➔ Sampling

Stochastic Mechanics & Optimization Lab
Differentiate both sides of PC form of response with basis random variables, according to multi-indices.

Multi-indices: \( \binom{d}{n} = \binom{d+n-1}{n} \)

\[
\frac{\partial m^{(k)}}{\partial \xi_1^{m_1^{(k)}} \partial \xi_2^{m_2^{(k)}} \ldots \partial \xi_d^{m_d^{(k)}}} \left| \xi_1, \xi_2, \ldots, \xi_d = 0 \right.
\]

\[
\partial m^{(k)} f(\sum_{j=0}^{d} \lambda_j \psi_j(\xi_1), \ldots, \sum_{j=0}^{d} \lambda_j \psi_j(\xi_d))
\]

\[
= \frac{\partial \xi_1^{m_1^{(k)}} \partial \xi_2^{m_2^{(k)}} \ldots \partial \xi_d^{m_d^{(k)}}}{\partial \xi_1^{m_1^{(k)}} \partial \xi_2^{m_2^{(k)}} \ldots \partial \xi_d^{m_d^{(k)}}} \left| \xi_1, \xi_2, \ldots, \xi_d = 0 \right.
\]

P+1 multi-indices gives linear system in the form: \( Ax=b \)

Evaluate at \( \xi_1, \xi_2, \ldots, \xi_d = 0 \) or mean values of RV.

Obtain sensitivities using ModFFD and find \( a_k \).
Methodology of PCDD

START

Write function $f$ and input variables in polynomial chaos expansion form

$$y = f(x_1, x_2, \ldots, x_d)$$

$$\sum_{i=0}^{P} a_i \Psi_i(\xi) = f\left(\sum_{i=0}^{P} v_i \psi_i\right)$$

Differentiate $\sum_{i=0}^{P} a_i \Psi_i(\xi)$ successively

Evaluate at $\xi_1, \xi_2, \ldots, \xi_d = 0$ to get matrix $A$ and $x = (a_0, a_1, a_2, \ldots, a_d)$ of linear system $Ax = b$

Find sensitivities $f^m(\xi_1, \xi_2, \ldots, \xi_d)$, in column vector $b$

Substitute matrix $A$ and column vector $b$ in $Ax = b$

Solve for coefficients $a_i$'s

$(i = 0, 1, 2, 3, \ldots, P)$

Substitute $a_k$ in chaos expansion expression to get approximation of a function

STOP
- Number of Samples for PCDD:
  \[ P + 1 = \binom{n + d}{n} = \frac{(n+d)!}{n!d!} \]
- For Quadrature, exponential increase with increase in dimension size → “Curse of Dimensionality”
- For Stochastic Collocation and PCDD faster rate of convergence
Small Sampling Domain

PCDD (136)

Large Sampling Domain

Tensor Points (256)

Comparison of Samples for $d=2$ and $n=15$
Simple Example

\[ Y = X^2 \]

where \( X \) is Normally distributed random variable with mean and standard deviation as \( x_0 \) and \( x_1 \), respectively.

\[ Y = y_0 + y_1 \xi + y_2 (\xi^2 - 1) \quad X = x_0 + x_1 \xi \]

Using differentiation technique,

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\begin{bmatrix}
y_0 \\
y_1 \\
y_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
x_0^2 \\
2x_0x_1 \\
2x_1^2 \\
\end{bmatrix}
\]

- No approximation involved
- PC order of response should depend upon the relationship between input variables and response, not on the magnitude of uncertainty in input variables.
- As order of PC increases, the condition number of PC coefficient matrix increases
Higher Order Forward Finite Difference (ModFFD), uses Taylor Series expansion up to order $n$

$$f(x_1 + m_1^{(k)}h_1, \ldots, x_d + m_d^{(k)}h_d) = \sum_{j=0}^{p} \frac{f^{m(j)}(x)}{m(j)!} \prod_{i=1}^{d} (m_i^{(k)}h_i)^{m_i} + O(h^{n+1})$$

Multi-indices represent differentiation order, $|m^{(k)}|$

$$f^{m^{(k)}}(m_1^{(k)}, \ldots, m_d^{(k)})(\vec{x}) = \left. \frac{\partial^{m^{(k)}} f}{\partial \xi_1^{m_1^{(k)}} \ldots \partial \xi_d^{m_d^{(k)}}} \right|_{\xi_1 = 0 \ldots \xi_d = 0} \left( \sum_{j=0}^{t} \lambda_j \psi_j(\xi_1), \ldots, \sum_{j=0}^{t} \lambda_j \psi_j(\xi_d) \right)$$
- Change step-sizes \((m_i^{(k)}h_i)\) using multi-indices to obtain \(P+1\) Taylor Series expansion equations and \(C*y=d\).
- Unknown sensitivities are in vector, \(y\).
- Obtain response values(samples), \(d\) for different realization of random input variables.
- The \((P+1) \times (P+1)\) matrix, \(C\) contains components:

\[
C(p, q) = \prod_{i=1}^{d} \frac{\left(\Delta h_i \ast m_i^{(p)}\right)^{m_i^{(q)}}}{(m_i)^q!}
\]

- Solve to get \(P+1\) sensitivities, \(f^{(m)}\).
ModFFD has order of accuracy $O(h^{n-|m|+1})$ whereas FOFFD has $O(h)$.

- For small sum $|m|$, accuracy is high.
- For large sum $|m|$, accuracy is similar to First Order Forward Finite Difference.
- When chaos order increased, accuracy also increases.
\[ y = e^x \]

\( X \) is normally distributed RV with mean and standard deviation as 2. Hence, \( Y \) becomes Log-normal random variable

Exact values:

<table>
<thead>
<tr>
<th>Mean</th>
<th>54.59812</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variance</td>
<td>159773.8</td>
</tr>
<tr>
<td>CoV</td>
<td>7.3211</td>
</tr>
</tbody>
</table>

95\% confidence bounds for variance with 1e10 samples: [159769.4049, 159778.2621]
Convergence obtained when chaos order, $n \geq 15$ in Mean and $n \geq 19$ in Std for PCDD

- ModFFD2 and FOFFD2 gave better results
- ModFFD2 has accuracy close to Analydiff
- However, Std obtained with Stochastic Collocation has large error and didn’t converge until order $n=22$
Sensitivity values obtained for order $n=22$

<table>
<thead>
<tr>
<th>$k^{th}$ sensitivity</th>
<th>Sensitivities</th>
<th>% Absolute Error in Sensitivities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analydiff</td>
<td>ModFFD2</td>
</tr>
<tr>
<td>1</td>
<td>14.7781</td>
<td>14.7781</td>
</tr>
<tr>
<td>2</td>
<td>29.5562</td>
<td>29.5562</td>
</tr>
<tr>
<td>3</td>
<td>59.1124</td>
<td>59.1124</td>
</tr>
<tr>
<td>20</td>
<td>3.8739e+06</td>
<td>3.8747e+06</td>
</tr>
<tr>
<td>21</td>
<td>7.7479e+06</td>
<td>7.7029e+06</td>
</tr>
<tr>
<td>22</td>
<td>1.5496e+07</td>
<td>1.7213e+07</td>
</tr>
</tbody>
</table>

Accuracy $O(h^{22})$ for $1^{st}$ Order Sensitivity

Accuracy $O(h^1)$ for $22^{nd}$ Order Sensitivity
PDF for 1-D Problem

PDF for $y = e^x$

- ModFFD2 and FOFFD2 PDF plots are the best among ModFFD and FOFFD.
- Overall, ModFFD2 follows same pattern and is close to Analydiff
- No Convergence with Stochastic Point-Collocation

Chaos Order, $n=22$

<table>
<thead>
<tr>
<th>Method</th>
<th>% Error of Mean</th>
<th>% Error of Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analydiff</td>
<td>8.307e-4</td>
<td>0.131</td>
</tr>
<tr>
<td>ModFFD2</td>
<td>8.530e-4</td>
<td>0.118</td>
</tr>
<tr>
<td>FoFFD2</td>
<td>2.0311</td>
<td>8.302</td>
</tr>
<tr>
<td>COLL, $np=2$</td>
<td>41.005</td>
<td>52.652</td>
</tr>
</tbody>
</table>
Two-Dimensional Problem

\( y = \log(1 + x_1^2) \times \sin(5x_2) \)

\( x_1 \) and \( x_2 \) are uniformly distributed with mean 2.0 and PDF1 and PDF2 height = 0.722

<table>
<thead>
<tr>
<th>( f(x_1, x_2) )</th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Exact</strong></td>
<td>0.0791</td>
<td>1.1241</td>
</tr>
<tr>
<td><strong>MC</strong></td>
<td>0.0778</td>
<td>1.1260</td>
</tr>
<tr>
<td>95% CI of MC</td>
<td>[0.0708, 0.0852]</td>
<td>[1.1229, 1.1289]</td>
</tr>
</tbody>
</table>
- ModFFD1 with large step-size converged at very high order and FOFFD1 has large error
- Onwards order 10, error almost zero for Analydiff, ModFFD2, ModFFD3, FOFFD3
- Stochastic Collocation and ModFFD2 have equivalent performance
For $d=2$ and $n=15 \Rightarrow 136$ terms

- ModFFD has mostly negligible error
- Few sensitivities of ModFFD with order $n=15$, have error $\geq 5\%$, similar to FOFFD

PDF plots of the Analydiff, ModFFD2, and Colloc2 are similar to LHS simulations
Uncertainties exist at different levels of Composite
Accumulates as we go higher level
Affects overall performance and reliability
Uncertainty type: - I). Epistemic, due to inherent randomness in input parameters and II). Aleatory, due to uncertainties in mathematical models
Prevalent methods for UQ in Composites: Sampling techniques and Perturbation approaches
Spectral approaches with higher accuracy are being used more recently
Epistemic Uncertainties in Composites

- Class AB captures uncertainties at different scales, so modeling generally preferred at meso-scale
Uncertainties considered at ply-level

Two cases studied:

I) Uncertain material properties (4 RV)
II) Uncertain material and geometric properties (20 RV)

Deterministic model is Eight layered, simply-supported plate under uniform pressure on top surface (11,000 Pa)

MSC.NASTRAN used as *black box* to generate response for samples

PCDD Samples are generated using step-sizes:

\[
\frac{\Delta x_i}{x_{i,\text{mean}}} \approx 10^{-2}
\]

LHS simulation, PCDD, and Stochastic Point-Collocation (COLL) used to obtain stochastic responses,
## Material and Geometric Properties of CFRP laminate

<table>
<thead>
<tr>
<th>Type</th>
<th>Random Variables</th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Material Properties</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_{11} \ (GPa)$</td>
<td></td>
<td>142.21</td>
<td>7.366</td>
</tr>
<tr>
<td>$E_{22} \ (GPa)$</td>
<td></td>
<td>8.555</td>
<td>0.343</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td></td>
<td>0.251</td>
<td>0.091</td>
</tr>
<tr>
<td>$G_{12} \ (GPa)$</td>
<td></td>
<td>4.376</td>
<td>0.216</td>
</tr>
<tr>
<td><strong>Geometric Properties</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_i \ (mm), i = 1,2,\ldots,8$</td>
<td></td>
<td>4.7</td>
<td>0.5</td>
</tr>
<tr>
<td>$\theta_1, \theta_3, \theta_6, \theta_8 \ (deg)$</td>
<td></td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$\theta_2, \theta_4, \theta_5, \theta_7 \ (deg)$</td>
<td></td>
<td>90</td>
<td>10</td>
</tr>
<tr>
<td><strong>Plate Dimensions</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length \ (a=1.7 m), &amp; Width \ (b=1.5 m)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- All input random variables have **Gaussian Distribution**
- Assume Mean values of Geometric properties for **4 RV** problem
Mesh of a Composite Laminate

- Transverse displacements calculated at Grid Points A, B, and C ($y_i, i = 1, 2, 3$)
- In-plane stresses calculated for top and bottom layer of Element P and Element S ($y_i, i = 4, ..., 15$)
- Effective Material Properties \((E_1, E_2, \nu_{12}, G_{12})\) as Uncertain inputs
- For all the 15 responses, PCDD has negligible error in Mean and less error in Std
- Stochastic Point-Collocation did not yield accurate results even with chaos order 10 (using 2,002 LHS samples)
Effective Material Properties \((E_1, E_2, \nu_{12}, G_{12})\) and Geometric Properties \((\theta_i, t_i)\) are uncertain \(\Rightarrow 20\) random inputs

<table>
<thead>
<tr>
<th>Response</th>
<th>LHS simulations</th>
<th>PCDD, (n=2)</th>
<th>COLL, (n=3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>Mean</td>
</tr>
<tr>
<td>(w_A)</td>
<td>-1.533</td>
<td>0.195</td>
<td>-1.551</td>
</tr>
<tr>
<td>(w_B)</td>
<td>-0.796</td>
<td>0.102</td>
<td>-0.804</td>
</tr>
<tr>
<td>(w_C)</td>
<td>-1.101</td>
<td>0.139</td>
<td>-1.114</td>
</tr>
<tr>
<td>(\sigma_{\text{max}})(\big</td>
<td>_{S_t})</td>
<td>-0.555</td>
<td>0.158</td>
</tr>
<tr>
<td>(\sigma_{\text{min}})(\big</td>
<td>_{S_t})</td>
<td>-6.458</td>
<td>1.770</td>
</tr>
<tr>
<td>(\tau_{\text{max}})(\big</td>
<td>_{S_t})</td>
<td>2.951</td>
<td>0.941</td>
</tr>
</tbody>
</table>

\(n_{\text{samples}}\) for best results \(\Rightarrow 50,000\), \(231\), \(3,542\)

Units: \(w\) (\text{mm}) and stresses (\text{MPa})
Subscripts for Stresses

t: Top Layer
b: Bottom Layer
Macroscale Modeling: Problem 2

[Graphs and data related to stress distributions and PDFs]

Stochastic Mechanics & Optimization Lab
For PCDD, responses $y_i, i=4-9$ required $n=2$ (231 samples) while remaining responses required $n=3$ (1,771 samples).

Transverse displacements with COLL are more accurate than PCDD whereas in-plane stresses with PCDD have less error in both Mean and Std.

With COLL less accurate solution obtained for stresses even with $n=3$ (3,542 samples) and higher error with $n=4$ (21,252 samples).
For transverse displacements and stresses, less than 2% error in mean.

Displacements have max. error of almost 3% and stresses have max. error of 5% in standard deviation.

With Stochastic Collocation only the transverse displacements are accurate.

Using 231 samples, provided similar results to 50,000 LHS simulations and more accurate than Stochastic Collocation.
Uncertainties considered starting from constituents (fiber and matrix) level that results in uncertain Material properties and uncertain responses

Geometric Uncertainties also considered

UQ for multiscale can be performed in two ways:

i. First obtain stochastic model for Material properties using PCDD with uncertain constituents inputs, then obtain stochastic responses

ii. Directly obtain stochastic model for responses without finding stochastic model for Material properties

First method is preferred in Multiscale analysis for use with commercial software, like MSC.NASTRAN
Uncertainties in fiber and matrix properties considered ($E_{11}^f, E_{22}^f, E^m, V_f, \nu_{12}^f, \nu^m, G_{12}^f$)

Random inputs are represented using PC expansion, for instance,

$$E_{11}^f = \sum_{j=0}^{n_1} \lambda_j \psi_j (\xi_1)$$

Existing micromechanics models can be used e.g.:- Voight, Reuss, Halpin Tsai, Method of Cells

Highly accurate Halpin Tsai model used

Instead of micromechanics model, FEM can be used, but computationally expensive
Effective Material Properties of a Lamina considered as random, \( E_1, E_2, \nu_{12}, G_{12} \)

Longitudinal Modulus of Elasticity, \( (E_1) \)

\[
E_1 = E_{11}^f V_f + E^m (1-V_f) ; \quad \text{and} \quad E_1(\xi_1, \xi_3, \xi_4) = \sum_{k=0}^{P_1} E_{1k} \psi_k(\xi_1, \xi_3, \xi_4)
\]

Transverse Modulus of Elasticity, \( (E_2) \)

\[
E_2 = E_m \left( \frac{1+\xi \eta V_f}{1-\eta V_f} \right) ; \quad \text{where} \quad \eta = \frac{\left( \frac{E_{2f}}{E_m} \right)^{-1} - \xi}{\left( \frac{E_{2f}}{E_m} \right) + \xi}
\]

\( \xi \) :- curve fitting parameter dependent on fiber packing arrangement

where \( \xi = 2 \) (for \( E_2 \), while using circular fibers)

\[
E_2(\xi_2, \xi_3, \xi_4) = \sum_{k=0}^{P_2} E_{2k} \psi_k(\xi_2, \xi_3, \xi_4)
\]
Major Poisson’s Ratio, \((v_{12})\)

\[ v_{12} = v_{12}^f V_f + v^m (1 - V_f); \quad \text{and} \quad v_{12} (\xi_4, \xi_5, \xi_6) = \sum_{k=0}^{P^3} v_{12k} \psi_k (\xi_4, \xi_5, \xi_6) \]

Shear Modulus of Elasticity, \((G_{12})\)

\[ G_{12} = G_m \left( \frac{1 + \xi \eta V_f}{1 - \eta V_f} \right); \quad \text{and} \quad \eta = \frac{G_f}{G_m} - 1 \]

\[ \eta = \frac{G_f}{G_m} + \xi \]

where \(\xi = 1\). Matrix is isotropic, \(G_m = \frac{E_m}{2(1 + \nu_m)}\)

Hence,

\[ G_{12} (\xi_3, \xi_4, \xi_6, \xi_7) = \sum_{k=0}^{P^4} G_k \psi_k (\xi_3, \xi_4, \xi_6, \xi_7) \]
Performance behavior of Composite depends on stochastic effective material properties and geometric properties

Stochastic response can be represented as:-

\[ f = f \left( E_{11}^f, E_{22}^f, E_m, V_f, \nu_{12}^f, \nu_m, G_{12}^f, t_1, \ldots, t_8, \theta_1, \ldots, \theta_8 \right) \]

\[ = f \left( \xi_1, \xi_2, \ldots, \xi_7, \xi_8, \ldots, \xi_{15}, \xi_{16}, \ldots, \xi_{23} \right) \]

The constitutive equation using CLPT is:-

\[
\begin{bmatrix}
N(\bar{\xi}) \\
M(\bar{\xi})
\end{bmatrix} = 
\begin{bmatrix}
A(\bar{\xi}) & B(\bar{\xi}) \\
B(\bar{\xi}) & D(\bar{\xi})
\end{bmatrix}
\begin{bmatrix}
\varepsilon^0(\bar{\xi}) \\
\kappa(\bar{\xi})
\end{bmatrix}
\]

However, PCDD uses deterministic codes (here **MSC.NASTRAN**) as black box and requires no modification of constitutive equations.
Multi-scale Modeling Framework of CFRP Composite Laminate

1. **Start**

2. Identify all the input random variables and outputs. Represent all the input RV’s in PC form

3. Choose a micromechanics model. Use PCDD to obtain stochastic effective material properties

4. Generate different values for all RV’s according to multiindices

5. Expand the output function in PC form

6. Use PCDD to obtain the PC expansion coefficients and statistics of response

7. **End**
Eight layered, simply-supported edges under uniform pressure on top surface (11,000 Pa)

MSC.NASTRAN used with 20 × 20 QUAD4 2D elements

All 23 input random variables have Gaussian Distribution

<table>
<thead>
<tr>
<th>Type</th>
<th>Random Variables</th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constituents Material Properties</td>
<td>$E_{11}^f$ (MPa)</td>
<td>$2.137 \times 10^5$</td>
<td>$1.034 \times 10^5$</td>
</tr>
<tr>
<td></td>
<td>$E_{22}^f$ (MPa)</td>
<td>$1.380 \times 10^5$</td>
<td>690.000</td>
</tr>
<tr>
<td></td>
<td>$E^m$ (MPa)</td>
<td>0.200</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td>$\nu_{12}^f$</td>
<td>0.251</td>
<td>0.091</td>
</tr>
<tr>
<td></td>
<td>$\nu^m$</td>
<td>0.350</td>
<td>0.017</td>
</tr>
<tr>
<td></td>
<td>$V_f$</td>
<td>0.660</td>
<td>0.013</td>
</tr>
<tr>
<td></td>
<td>$G_{12}^f$ (MPa)</td>
<td>$1.380 \times 10^4$</td>
<td>690.000</td>
</tr>
<tr>
<td>Geometric Properties</td>
<td>$t_i$ (mm), $i = 1, 2, \ldots, 8$</td>
<td>4.7</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>$\theta_1$, $\theta_3$, $\theta_6$, $\theta_8$ (deg)</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>$\theta_2$, $\theta_4$, $\theta_5$, $\theta_7$ (deg)</td>
<td>90</td>
<td>10</td>
</tr>
<tr>
<td>Plate Dimensions</td>
<td>Length (a=1.7 m), &amp; Width (b=1.5 m)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Constituents material properties and geometric properties are random
Four Stochastic Material Properties model

- ModFFD1, ModFFD2, and ModFFD3 represents the PCDD using ModFFD with step-sizes \((h=1e^{-1}, h=1e^{-2}, h=1e^{-3})\). Compared with \(1e+06\) LHS simulations.

- Convergence with Order 2 and 10 Samples
- Convergence with Order 2 and 10 Samples
Convergence with Order 4 and 35 Samples

Convergence with Order 5 and 56 Samples

<table>
<thead>
<tr>
<th>Method</th>
<th>$E_1$ (GPa)</th>
<th>$E_2$ (GPa)</th>
<th>$\nu_{12}$</th>
<th>$G_{12}$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>Mean</td>
<td>Std</td>
</tr>
<tr>
<td>PCDD</td>
<td>142.215</td>
<td>7.366</td>
<td>8.555</td>
<td>0.343</td>
</tr>
<tr>
<td>LHS</td>
<td>142.215</td>
<td>7.356</td>
<td>8.555</td>
<td>0.343</td>
</tr>
<tr>
<td>% Error</td>
<td>3.73e-4</td>
<td>1.60e-1</td>
<td>6.40e-5</td>
<td>3.48e-2</td>
</tr>
</tbody>
</table>

Highly Accurate
- Stochastic Effective material properties along with Uncertain Geometric Properties considered
- Transverse displacements at grid points
- In-plane stresses of the elements calculated at top and bottom layer

20 × 20 QUAD4 2D elements

The PDFs match well
Multiscale Modeling: Results

Stochastic Mechanics & Optimization Lab
PCDD with PC order two required 300 samples

300 PCDD samples Vs 50,000 LHS simulations

<table>
<thead>
<tr>
<th>Response</th>
<th>PCDD</th>
<th>LHS simulations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
</tr>
<tr>
<td>$w_A$</td>
<td>-1.555</td>
<td>0.200</td>
</tr>
<tr>
<td>$w_B$</td>
<td>-0.803</td>
<td>0.105</td>
</tr>
<tr>
<td>$w_C$</td>
<td>-1.114</td>
<td>0.142</td>
</tr>
<tr>
<td>$\sigma_{\max}|_{s_b}$</td>
<td>8.822</td>
<td>0.859</td>
</tr>
<tr>
<td>$\tau_{\max}|_{s_b}$</td>
<td>4.044</td>
<td>0.395</td>
</tr>
<tr>
<td>$\sigma_{\max}|_{p_b}$</td>
<td>11.869</td>
<td>1.155</td>
</tr>
<tr>
<td>$\tau_{\max}|_{p_b}$</td>
<td>4.044</td>
<td>0.395</td>
</tr>
</tbody>
</table>

Units: $w$ (mm) and stresses (MPa)
PDFs of PCDD and LHS simulations are in close agreement for all responses.

Using only 300 PCDD samples provided similar accuracy to 50,000 LHS simulations.

Higher accuracies observed in mean as compared to standard deviation.

Transverse displacements have higher accuracies than in-plane stresses.

Transverse displacements have max. error of 1.2% and 2.1% in Mean and Std, respectively.

In-planes stresses have error of 0.26%~3.65% and 0.45%~5.09% in Mean and Std.
Conclusion and Future Work

- PCDD is a new Non-Intrusive technique for UQ
- ModFFD is a higher accuracy method for Sensitivity Calculation and is used in PCDD
- The number of samples required is very less (equal to number of polynomials) than existing non-intrusive methods
- Very high accuracy results obtained for analytical problems
- Results of CFRP composite laminate problems with uncertainties at different scales demonstrated satisfactory results with huge computational savings
- For response with higher order of magnitudes, PCDD converged faster and accurately than Stochastic Point-Collocation
- PCDD and Stochastic Point-Collocation have similar accuracy in some cases
Conclusion and Future Work

- Improve the Computational Efficiency even further
- Improve the accuracy by increasing the number of Samples
- Dimension Adaptive PC, Sparse PC, and Adaptive Sampling
- Implementation of PCDD in Robust Optimization
Thank You !!!